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Routh vol II Applications
Kirchoff Math. Phys.

John C. Bird,
Ann Arbor,
Michigan,
February 27, 1895,

George K. Lawton,
G. S. State.

Ann Arbor Feb. 27, '94.

Frances Lawton

AN ELEMENTARY TREATISE

ON

DYNAMICS.

FIFTH EDITION.

AN ELEMENTARY TREATISE

ON

THE INTEGRAL CALCULUS,

CONTAINING

APPLICATIONS TO PLANE CURVES AND SURFACES.

BY

BENJAMIN WILLIAMSON, F. R. S.

SIXTH EDITION.

AN ELEMENTARY TREATISE

ON

THE DIFFERENTIAL CALCULUS,

CONTAINING

THE THEORY OF PLANE CURVES.

BY

BENJAMIN WILLIAMSON, F.R.S.

AN ELEMENTARY TREATISE

ON

DYNAMICS,

CONTAINING

APPLICATIONS TO THERMODYNAMICS,

WITH

NUMEROUS EXAMPLES.

BY

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SECOND EDITION, REVISED AND ENLARGED.

NEW YORK:
D. APPLETON AND COMPANY.
1889.

PREFACE.

Although in recent years several important works on Dynamics have been published in England, yet none have been issued which seem to fill the *rôle* contemplated in this book. In its composition we have started from the most elementary conceptions, so that any Student who is acquainted with the conditions of Equilibrium and with the notation of the Calculus can commence the Treatise without requiring the previous study of any other work on the subject. The first half contains a tolerably full treatment of what is usually styled the Dynamics of a Particle. The latter half treats of the Kinematics and Kinetics of Rigid Bodies; and throughout we have kept the practical nature of the subject in view, and have, in general, avoided purely fancy problems.

In an early chapter we have introduced and elucidated the general principle of Work or Energy, and have given subsequently a more complete treatment of this great principle, illustrating it by a brief application to the theory of Thermodynamics. In the latter part of the book we have borrowed largely from Thomson and Tait's *Natural Philosophy;* Routh's *Rigid Dynamics;* Schell's *Theorie der*

Bewegung und der Kräfte; and Clausius' *Mechanical Theory of Heat*; our aim having been simply to enable the Student to acquire as easily as possible a knowledge of the subject of which we treat.

In this Edition we have carefully revised and to a considerable extent rearranged the entire Work. In doing so we have developed, and in some cases rewritten, many portions of the subject, more especially that on generalized coordinates in connexion with Lagrange's and Hamilton's methods. We have also exhibited the general theory of small oscillations in a new form, and one which we hope will be easily comprehended by the Student.

To those who desire to pursue the study of Dynamics to its highest development, the perusal of the great treatise of Thomson and Tait, as also that of Routh, will, we hope, be facilitated by using the present Work as an introduction.

We may add that to the latter writer our obligations, as the reader will find, have been largely increased in this Edition.

TRINITY COLLEGE,
May, 1889.

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CHAPTER I.

VELOCITY.

1. **Matter.**—We give the name of matter to that which exclusively occupies space, and which we regard as the permanent cause of any of our sensations. Portions of matter which are bounded in every direction are called bodies. Every body has necessarily a determinate volume, and an external form or surface; and exists, or is conceived to exist, in space.

A portion of matter indefinitely small in all its dimensions is called a material *particle*. Every body may be regarded as consisting of an indefinitely great number of particles. The name of *force* is given to any cause which produces, or tends to produce, motion in matter. The branch of Mechanics which treats of motion produced in a body by the action of force is commonly called Dynamics.

We commence with the consideration of motion in itself, without any regard to its cause.

2. **Motion, Velocity.**—When a body continually changes its position in space, it is said to be in motion; and the *rate* and the *direction* of the motion of any of its points at any instant is called the *velocity* of the point at that instant.

The motion of a point is said to be *rectilinear* or *curvilinear* according as its path is a right line or curved.

In the case of curvilinear motion, the *direction* of motion of a particle at any instant is that of the tangent to its path, drawn at the point occupied by the moving particle at the instant.

3. **Motion of Translation.**—If all the points of a rigid body move, at each instant, in parallel directions, the body is said to have a motion of translation only; and the motion of the body is completely determined when that of any one of its points is known. It is usual, in this case, to take its *centre of mass* as the point whose motion determines that of the body.

In our earlier chapters, whenever we speak of a rigid body moving, we suppose it to have a motion of translation solely, and we consider its path as that of its centre of mass.

4. **Uniform Motion, Velocity.**—If a point move over equal lengths or spaces,* in equal intervals of time, however short the intervals be taken, its motion is said to be uniform; and its velocity is measured by the *space described in the unit of time:* this is the same at every instant so long as the motion continues uniform.

A *second* is usually adopted as the unit of time; and, in this country, a foot as the unit of length. Thus, the velocity of a point which moves over five feet in each second is said to be a velocity of 5 feet per second, and is numerically denoted by 5; and similarly in other cases. If any other units of time and space be adopted, the number which represents the velocity of the moving point will have to be altered proportionally. Thus, we speak of a velocity of 10 miles an hour, or 100 yards a minute, &c.: each of these can be readily expressed in feet per second, when necessary.

The space, or length of the path described during any time, is usually denoted by the letter s, the velocity by v, and the time estimated in seconds by t.† In the case of *uniform motion*, the relation connecting these quantities can be immediately obtained. For, if the space described in one second be represented by v, that described in two seconds is represented by $2v$, that in three seconds by $3v$, and that in any number (t) of seconds by vt.

---

* The word *space* is employed in abbreviation for *length of path* described.

† Unless the contrary be stated, we shall in all cases assume a foot and a second as our units of space and time, *i.e.* we shall regard t as representing a *number* of seconds or parts of a second, and s as a number of feet.

Accordingly we have in the case of uniform motion the relation

$$s = vt. \tag{1}$$

This formula evidently holds good whatever be the *units* of space and time, and introduces the unit of velocity as that of a unit of space described in a unit of time. It is true for uniform curvilinear, as well as rectilinear motion; and also whether t represents a number of seconds, or any part of a second, however small.

Again, if s' denote the space described in the time t', we have $s' = vt'$, and hence

$$v = \frac{s' - s}{t' - t};$$

or the velocity, when uniform, is measured by the *space described during any interval of time divided by the number by which that time is represented.*

This result equally holds good if we suppose the interval of time, denoted by $t' - t$, to become *indefinitely* small; in which case the *limiting value* of $\frac{s' - s}{t' - t}$ or $\frac{ds}{dt}$ will still represent the velocity v.

EXAMPLES.

1. If a body, moving uniformly, pass over 10 miles in an hour, find its velocity in feet per second. *Ans.* $14\frac{2}{3}$,

2. If a body, moving uniformly with a velocity of 16 feet per second, pass over 100 miles, find the time of its motion. *Ans.* 9 hrs. 10 min.

3. Assuming that light travels from the sun to the earth in $8^m\ 30^s$, and that its velocity is 180,000 miles per second, calculate the distance of the sun. *Ans.* 91,800,000 miles.

4. If a velocity of 20 miles an hour be the unit of velocity, and a mile the unit of space, find the number which represents a velocity of 32 feet per second. *Ans.* $1\frac{1}{11}$.

5. Find in metres the velocity of a point on the earth's equator arising from the rotation of the earth on its axis. *Ans.* 463.

5. **Variable Motion.**—If the spaces described in equal intervals of time be not equal, the motion is said to be *variable*, and the velocity can no longer be measured by the space actually described in one second. The movable has, however, at each instant a certain definite velocity which is

measured by the space *which it would describe during a second, if it were conceived to move uniformly during that time with the velocity which it has at the instant under consideration.*

For example, when we say that a railway train is moving at the rate of 40 miles an hour, we mean that it would pass over 40 miles in the hour if it continued to move during that time with the speed which it has at the instant referred to.

Again, if we suppose that there are no *sudden* changes of velocity, the change in the velocity of a movable in any indefinitely small portion of time must be itself indefinitely small; as otherwise the velocity would not vary *continuously*. Accordingly, in such cases, we may suppose the motion as *uniform* during the indefinitely small time dt; and we shall have (as in the last Article) for the velocity v at any instant the equation

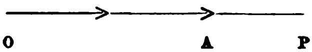
$$v = lim. \frac{s' - s}{t' - t} = \frac{ds}{dt}. \tag{2}$$

That is, in all cases the velocity of a point at any instant is measured by the limiting value of the space described in a small interval of time, divided by the number which represents that interval of time. This method of expressing velocity is sometimes concisely represented in the notation of Newton by the symbol $\dot{s}$.

6. **Mean Velocity.**—If a body describe the space s in the time t, then its *mean* or *average* velocity during that time is represented by $\frac{s}{t}$, being the velocity with which a body, moving uniformly, would describe the same space in the time t. The formula (2) can be immediately deduced from the consideration of mean or average velocity—for we may consider the velocity of a point at any instant as being its mean velocity during an infinitely small interval of time; whence we get, as before, the relation $v = \frac{ds}{dt}$.

7. **Geometrical Representation of a Velocity.**—Uniform rectilineal motion is completely determined when the direction and rate of motion are known. Hence the velocity of a point can be represented both in magnitude and direction by a right line.

Thus, if a point move uniformly in the line OP, so as to describe the space OA in the unit of time (one second suppose), the line OA may be taken to represent the velocity of the point both in magnitude and direction. The *arrow head* denotes the direction in which the motion takes place, namely from O to A.



This method of representation holds good also in the case of variable velocity, provided OA be the space which the body *would* describe in one second if its velocity remained unaltered in magnitude and direction (Art. 5).

In accordance with the principles established in Geometry, if the velocity of a particle moving from O to P be regarded as *positive*, velocity in the opposite direction, *i. e.* from P to O, must be regarded as *negative.*

8. **Kinematics.**—As our ideas of motion and velocity depend solely on our conceptions of space and time, the whole subject of motion admits of being treated as a branch of pure Mathematics; and, as such, has been discussed in many important treatises during recent years.

This branch of Mathematics is called Kinematics* (from *κίνημα*, motion), and in it the motion of a body is discussed without any reference to the force or forces by which the motion is produced. Questions of the latter class, *i. e.* of motion with reference to force, belong to the science of Dynamics, or what is now usually styled Kinetics.

The foregoing distinction should be observed by the student, as much indistinctness of conception arises from its not being carefully kept in mind in the study of Dynamics.

In the present treatise it is not proposed however to divide the treatment of the subject in the manner indicated, as to do so would require a complete discussion of motion (including rotation and kindred subjects) before entering on the most elementary problems in Dynamics. At the same time it will aid the student towards obtaining clear mechanical conceptions if he will consider what part of each problem

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* The name "Cinématique" was first given to this branch of Mathematics by Ampère, in his "Essai sur la philosophie des Sciences," 1834

discussed belongs properly to the science of Kinematics, and what to that of Dynamics or Kinetics.

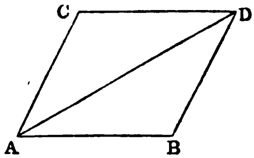
9. **Rest and Motion, Relative.**—We have defined rest and motion with reference to space. Now of space in itself or absolute space our senses take no cognizance, all that we perceive being matter or body as occupying or existing in space; but our senses give us no information as to whether any body occupies the same absolute position in space during successive intervals of time or not. Hence, of absolute rest we can have no perception or knowledge; and when we say that a body is at rest we mean that it does not alter its position with relation to other bodies which are considered fixed. For instance, bodies on the earth's surface are said to be at rest when they do not alter their position relatively to the earth's surface; we know however that the earth has at least two distinct motions, one of rotation relative to its axis; the other around the sun, regarded as fixed. As our idea of rest is only relative, so also must be our idea of motion: thus, a body is said to be in motion when it alters its position with respect to other bodies regarded as being at rest.

Hence all motions must be considered as relative: for instance, when we say that a body is moving at the rate of thirty miles an hour, we mean that such is its velocity relative to a place on the earth: its absolute velocity is immensely greater, and is obtained by combining this velocity with the absolute velocity of the earth itself.

Again, we speak of the same body as at *rest*, or as in *motion*, according as we compare its position with that of one object or of another. For example, a person seated in a railway carriage is said to be at rest relatively to the carriage, and to be in motion relatively to the earth, &c.

That a body may be regarded as having at the same instant two or more velocities is a matter of common experience: for instance, if a ball roll along the deck of a vessel, which is descending a river, we conceive the ball as having simultaneously one velocity along the deck; another, that of the vessel in the stream; a third, that of the river relatively to its banks, &c. The velocity of the ball, relatively to the earth, is got by compounding these separate velocities. We proceed to show in what manner this can be done.

10. **Composition of Velocities.**—Suppose a point to move uniformly, with a velocity v, along the line AB, while the line moves uniformly parallel to itself; then the point may be regarded as having the two velocities simultaneously. In order to find its position at the end of any time t, let AB be the space which it would describe in that time along AB considered as fixed; and let CD be the position of the moving line at the end of the same time; complete the parallelogram $ABDC$; then D will plainly be the position of the moving point at the end of the time t. Also, if v' be the uniform velocity of the point along the line AC, we shall have $AC = v't$, and $CD = vt$. Hence



$$\frac{AC}{CD} = \frac{v'}{v}.$$

Again, as this is independent of t, the ratio of AC to CD will be constant during the entire motion; and consequently the point will move from A to D along the diagonal AD.

To find the velocity of the moving point, we make $t = 1$ (or the unit of time) in the last; then AB and AC represent in magnitude and direction the component velocities of the moving point, and AD represents the resultant velocity: in other words, if a body be animated by two velocities represented in magnitude and direction by the sides of a parallelogram, the resultant velocity is represented in magnitude and direction by the diagonal of the parallelogram.

Conversely, any velocity may be regarded as equivalent to two velocities in any two directions, and the magnitudes of the component velocities can be determined by the preceding construction.

In like manner, if a body be animated simultaneously with three velocities, its resultant velocity is represented in magnitude and direction by the diagonal of the parallelepiped whose edges represent the component velocities. For we can compound two of these velocities by the method given above, and then compound their resultant with the third velocity. This principle can, plainly, be extended to the case of a point

supposed to be animated by any number of velocities simultaneously.

11. **Polygon of Velocities.**—It immediately follows that if a point be subjected to any number of simultaneous velocities its resultant velocity can be obtained by the following geometrical construction :—

From O, the original position of the point, draw OA, representing one of the given velocities in magnitude and direction; from A draw AB, parallel and equal to the line which represents a second velocity; and so on for the remaining velocities; then the line which connects O with the extremity of the line drawn parallel and equal to the line representing the last velocity will represent the resultant velocity, both in magnitude and direction.

This construction is called the polygon of velocity, and is in general a *gauche* polygon.

The preceding result admits of being stated otherwise, thus: If a body be subjected to two or more uniform velocities it will arrive at the same position at the end of any time as it would have arrived at if the several motions had taken place *successively* instead of simultaneously. This is adopted as an axiom by some writers on Mechanics, for it appears to be an immediate consequence of our ideas of motion. The student can easily see that the whole theory of the composition of velocities can be deduced from this principle.

12. **Component and Resultant Velocities.**—The velocities represented by AB and AC, in Art. 10, are called the *components* of the velocity represented by AD.

If a point describe a plane path, the usual method of representing its position is with reference to two fixed rectangular axes lying in the plane.

Then, if x, y be the coordinates of the moving point at any instant, its component velocities parallel, respectively, to the coordinate axes, are evidently, by Art. 5, represented by $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

Also, if a be the angle which the direction of motion at the instant makes with the axis of x, the component velocities are represented by $v \cos a$ and $v \sin a$, respectively; *i. e.* the velocity with which a point is moving in any fixed direc-

tion is equal to the component of its velocity in that direction.

Hence we get $v \cos a = \dfrac{dx}{dt}$, $v \sin a = \dfrac{dy}{dt}$. (3)

If we square and add, we get

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2; \therefore v = \frac{ds}{dt};$$

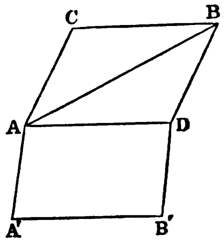
i.e. the velocity in a curvilinear path is represented in the same matter as in a rectilinear; this result might have been directly established from other considerations.

More generally, if x, y, z be the coordinates of a moving point at any instant, with reference to any system of coordinate axes, its component velocities parallel to the coordinate axes are plainly represented by $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$ and $\dfrac{dz}{dt}$, respectively. If the axes be rectangular, and if a, β, γ be the direction angles, and v the magnitude of the velocity of the point, then the component velocities parallel to the coordinate axes are represented by $v \cos a$, $v \cos \beta$, $v \cos \gamma$, respectively. Hence, in this case, we have

$$v \cos a = \frac{dx}{dt}, \quad v \cos \beta = \frac{dy}{dt}, \quad v \cos \gamma = \frac{dz}{dt}. \tag{4}$$

In Newton's notation, as in Art. 5, these component velocities are represented by the symbols, $\dot{x}$, $\dot{y}$, $\dot{z}$.

13. **Relative Velocity.**—If the point A be in motion along AB with a velocity represented by AB, and, at the same time, A' be in motion along $A'B'$ with a velocity represented by $A'B'$, to find their *relative* velocity.



Draw AD parallel and equal to $A'B'$, and construct the parallelogram $ACBD$; then the velocity AB may be regarded as equivalent to the velocities AD and AC; now the former velocity, being *equal and in the same direction as that of the other point A', will not alter the relative*

position of the points (Art. 10); consequently the latter component AC represents the relative velocity of the moving points, *i.e.* the velocity with which A is moving relatively to A', regarded as at rest.

Hence, to get the velocity of one moving point relatively to another which is also in motion, we suppose *equal and parallel motions given to both, each equal and opposite to the motion of the second point:* by this means that point is brought to rest, and the velocity of the other, relative to it, is had by compounding the new velocity with its original velocity.

14. **Components of Relative Velocity.**—Suppose (x, y, z), (x', y', z') to be the coordinates of the two moving points (M, M'), respectively, with reference to any coordinate system of fixed axes. Then, to get the motion of M', relatively to M, we suppose three axes drawn through M parallel, respectively, to the coordinate axes; and let ξ, η, ζ denote the coordinates of M, relative to these axes, and we have

$$\xi = x' - x, \quad \eta = y' - y, \quad \zeta = z' - z;$$

and hence

$$\frac{d\xi}{dt} = \frac{dx'}{dt} - \frac{dx}{dt}, \quad \frac{d\eta}{dt} = \frac{dy'}{dt} - \frac{dy}{dt}, \quad \frac{d\zeta}{dt} = \frac{dz'}{dt} - \frac{dz}{dt}; \qquad (5)$$

i.e.
$$\frac{dx'}{dt} - \frac{dx}{dt}, \quad \frac{dy'}{dt} - \frac{dy}{dt}, \quad \frac{dz'}{dt} - \frac{dz}{dt},$$

or
$$\dot{x}' - \dot{x}, \quad \dot{y}' - \dot{y}, \quad \dot{z}' - \dot{z},$$

represent the components of the relative velocity of the two moving particles.

EXAMPLES.

1. Two points are moving in rectangular directions, with velocities of 300 and 400 yards per minute; find their relative velocity in feet per second.

Ans. 25.

2. Two particles start simultaneously from different points, in given directions, with uniform velocities. Show how, by a geometrical construction, to determine the relative distance at the end of any time; and find when this distance is a minimum.

3. The tide is running out of the mouth of a harbour at the rate of $2\frac{1}{2}$ miles per hour; in what direction must a man, who can row in still water at the rate of 5 miles per hour, point the head of the boat in order to make for a point directly across the harbour?

4. A boat starts with a given velocity across a river; find the dirction in which she should steer, in order, without altering her course, to land at a given station at the opposite side of the river—the velocity of the stream, and also of the boat, being supposed known.

5. Two trains are moving, one due south, the other north-east. If their velocities be 25 and 30 miles an hour, respectively, calculate their relative velocity.

6. A railway train is moving at the rate of 30 miles an hour, when it is struck by a stone, moving horizontally and at right angles to the train with the velocity of 33 feet per second. Find the magnitude and direction of the velocity with which the stone appears to meet the train.

Ans. Resultant velocity is 55 feet.

Indian Civil Service Exam., 1876.

7. Two particles start simultaneously from A, B, two of the angular points of a square $ABCD$, in the directions AB, BC; and describe the periphery with constant velocities V, v, respectively, where V is greater than v, until one particle overtakes the other. Prove that the minimum distances between the particles occur at equal intervals of time, and that if $V : v :: m + 1 : m$, where m is an integer, the sum of all these minimum distances is

$$\frac{m(m+1)}{2\sqrt{m^2 + (m+1)^2}} \times \text{a side of the square.}$$

Camb. Math. Trip., 1871.

CHAPTER II.

ACCELERATION.

15. **Acceleration and Retardation of Motion.**—The velocity of a point is said to be accelerated or retarded according as it increases or diminishes with the time. This acceleration, or rate of change of velocity in a fixed direction, may be either uniform or variable. Retardation of motion is to be regarded as a *negative acceleration*, i.e. as an acceleration in the opposite direction to that of the motion.

16. **Uniform Acceleration.**—The motion of a point moving in a straight line is said to be uniformly accelerated when it receives *equal increments of velocity in equal times.* In this case the acceleration is measured by the additional velocity received in each unit of time. As a second is usually taken as the unit of time, we may define the acceleration of velocity in this case to be measured by the *additional velocity received by the movable in each second;* this acceleration is usually denoted by the letter f.

In the case of uniform acceleration in a right line we proceed to find expressions for the velocity at the end of any given time, and also for the space described.

17. **Velocity at any Instant.**—Let v_0 denote the velocity at the instant from which the time is reckoned; then, since the point receives in each second an additional velocity f, its velocity at the end of the first second is $v_0 + f$; at the end of the next second, $v_0 + 2f$; at the end of the third, $v_0 + 3f$; and at the end of n seconds, $v_0 + nf$. Or, if t denote the number of seconds in question, and v the velocity at the end of that time, we have

$$v = v_0 + ft. \tag{1}$$

If the point be supposed to start from rest, we have

$$v = ft;$$

that is, *the velocity acquired at the end of t seconds is t times that acquired at the end of one second.*

In the case of a uniformly retarded motion, f denotes the velocity *lost* in each second; and, if v_0 be the initial velocity, we shall have, as before, for the velocity at the end of t seconds,

$$v = v_0 - ft. \tag{2}$$

In this case the velocity becomes zero at the instant when $v_0 = ft$, or at the end of the time $\frac{v_0}{f}$. If the retardation continued afterwards, the velocity would become negative; that is, the point should proceed to move back in a direction opposite to that of its former motion.

It will be observed that the formulæ (1) and (2) differ only in the sign of f; they may accordingly be regarded as comprised in the same general formula, in which a retardation, as stated before, is regarded as a negative acceleration.

Examples.

1. If a body start from rest with a uniform acceleration of 7 feet per second, find its velocity at the end of three minutes.

Ans. 1260 feet.

2. In what time would a body acquire a velocity of 100 feet per second if it start from rest with a uniform acceleration of 32 feet per second?

Ans. $3\frac{1}{8}$ seconds.

3. A body starts from rest with the velocity of 1000 feet per second, and its motion is uniformly retarded by a velocity of 16 feet each second; find when it would be brought to rest.

Ans. 1 m. $2\frac{1}{2}$ sec.

4. A velocity of one foot per second is changed uniformly in one minute to a velocity of one mile per hour. Express numerically the rate of change of velocity when a yard and a minute are taken as the units of space and time.

Ans. $\frac{28}{3}$.

18. **Space described in any Time.**—To find the space described in any time in the case of uniform acceleration in a straight line.

From equation (2) we get

$$\frac{ds}{dt} = v = v_0 + ft;$$

hence, by integration,

$$s = v_0 t + \tfrac{1}{2} ft^2;$$

no constant being added since the space is measured from the position of the point when $t = 0$.

If the point start from rest we have

$$s = \tfrac{1}{2}ft^2.$$

In the case of uniformly retarded motion we have

$$s = v_0 t - \tfrac{1}{2}ft^2.$$

This and the preceding formula are represented by the single expression

$$s = v_0 t \pm \tfrac{1}{2}ft^2, \qquad (3)$$

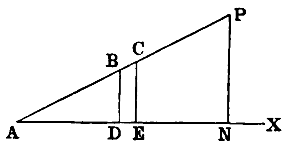
in which the upper or lower sign is given to f, according as the acceleration has place in the positive or negative direction.

Similarly, equations (1) and (2) are combined in the statement

$$v = v_0 \pm ft. \qquad (4)$$

The preceding result admits also of being established geometrically in the following manner, as given by Newton :—

Suppose the point to start from rest, and on any right line AX take portions AD, AE, &c., proportional to the intervals of time from the commencement of the motion, and erect perpendiculars DB, EC, &c., representing the corresponding velocities; then since the velocity at the end of any time (Art. 18) is proportional to that time, the ordinates BD, CE, &c., will be to one another in the same ratio as the times, *i. e.* as AD, AE, &c.; and consequently the points A, B, C, &c., all lie on a right line.



Again, let $AD = t$, $DE = \Delta t$, $BD = v$; then the space described in the infinitely small time Δt will be represented by $v\Delta t$, i. e. by the area $BDEC$; and accordingly the whole space described in the time represented by AN will be represented by the sum of the elementary areas, $BDEC$, &c., or by the whole area, APN, i. e. by $\tfrac{1}{2}AN \times PN$, or by $\tfrac{1}{2}vt$; therefore $s = \tfrac{1}{2}ft^2$, as before.

If the point be supposed to start with an initial velocity

v_0, the student will find no difficulty in supplying the corresponding construction.

19. **Relation between Velocity and Space.**—If we eliminate t between equations (3) and (4), we get

$$v^2 = v_0^2 \pm 2fs, \tag{5}$$

in which the upper or lower sign is taken according as the acceleration is in the direction of the motion or in the opposite direction.

We shall resume the consideration of these equations when we come to the investigation of the motion of a body under the action of a constant force.

20. **Algebraic Expression for an Acceleration.**—In the case of a point moving with a uniform acceleration, let v represent the velocity at the end of the time t, and v' that at the time t'; then by (1) we have

$$v = v_0 + ft, \quad v' = v_0 + ft',$$

and hence
$$f = \frac{v' - v}{t' - t}.$$

Moreover, since this result holds, however small the interval of time represented by $t' - t$ may be, we have, as in Art 4,

$$f = \frac{dv}{dt}.$$

21. **Variable Acceleration.**—In the case of the motion of a point in a right line, if the acceleration is not uniform, but varies continuously according to any law, we plainly (as in Art. 5) may suppose that the motion is uniformly accelerated during an infinitely small time dt; or (which is the same thing) that the acceleration at any instant is measured by what the *increase of velocity in a unit of time would have been if its rate of increase had been uniform during that time, and the same as that at the instant in question.* Hence the acceleration at any instant is defined as *the rate of change of the velocity at that instant*, and is measured in all cases by the ratio of the increment of the velocity at the instant to the increment of the time.

Accordingly we have, whether the acceleration be uniform or variable, the relations

$$f = \frac{dv}{dt} = \frac{d^2s}{dt^2}. \tag{6}$$

These are expressed in Newton's notation in the form

$$f = \dot{v} = \ddot{s}.$$

All these results apply equally to the case of retardation of motion, which is always to be regarded as a negative acceleration.

22. **Geometrical Representation of an Acceleration.**—From the preceding it appears that the acceleration of the motion of a point, whether it be uniform or variable, is in all cases measured by a velocity. Hence it can be *represented, both in magnitude and direction, by a right line,* in the same manner as velocity (Art. 7).

Hence, also, we may regard a point as receiving two or more simultaneous accelerations of motion, and can determine the resultant acceleration by a geometrical construction, as in Arts. 10 and 11.

Consequently, accelerations are compounded and resolved according to the same laws as velocities.

23. **Component Accelerations Parallel to Fixed Axes.**—If x, y, z denote the coordinates relative to a fixed rectangular system of axes, of the position of a moving point at the end of the time t; then, as in Art. 12, its component velocities parallel to the axes of coordinates are represented by $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, respectively.

Hence, since the acceleration of motion in any direction is measured by the rate of change of the velocity in that direction, we have for the accelerations parallel to the axes of coordinates the expressions

$$\frac{d\left(\frac{dx}{dt}\right)}{dt}, \quad \frac{d\left(\frac{dy}{dt}\right)}{dt}, \quad \frac{d\left(\frac{dz}{dt}\right)}{dt},$$

or

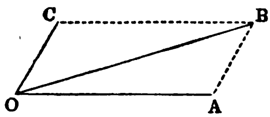
$$\ddot{x} = \frac{d^2x}{dt^2}, \quad \ddot{y} = \frac{d^2y}{dt^2}, \quad \ddot{z} = \frac{d^2z}{dt^2}, \tag{7}$$

where, in accordance with Newton's notation, $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ denote the accelerations parallel to the axes of x, y, z, respectively. The total acceleration of the motion of the point is the resultant of these accelerations.

It is plain that this acceleration is independent of any previously existing velocity, which may or may not be *in the same direction*.

The question of acceleration in curvilinear motion can also be treated in another manner, as follows:—

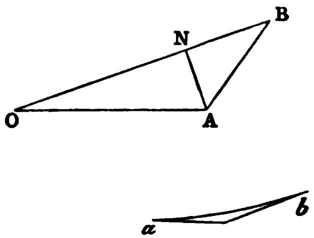
24. **Curvilinear Motion, Change of Velocity, Total Acceleration.**—Suppose a point to move in a curvilinear path, and from any point O let the line OA be drawn, representing in magnitude and direction the velocity of the moving point at any instant. Let OB, in like manner, represent its velocity at the end of the interval of time Δt. Join AB, and complete the parallelogram $OABC$. Then the velocity represented by OB is equivalent to the component velocities represented by OA and OC; but if the velocity of the point had not changed during the interval Δt, it would have been represented by OA; hence OC, or AB, *represents in magnitude and direction the change of velocity in the time* Δt.



Again, since the acceleration of the velocity of a movable, at any instant is, in all cases, measured by the *rate of change of the velocity* for that instant, it follows, as in (5), that if we regard the interval of time Δt as becoming infinitely small, the acceleration of the motion is represented by the limiting value of $\frac{AB}{\Delta t}$. This limiting value is called the *total acceleration* of the motion of the particle at the instant.

25. **Tangential and Normal Accelerations.**—Again, suppose a to denote the position of the moving point at the end of the time t, and b its position after a small interval of time, Δt, and draw tangents to the path at the points a and b. Also, as before, from any point O draw OA, OB parallel to these tangents, and representing the velocities

at a and b, respectively. Then, by the preceding Article, AB represents the total change in the velocity in the interval Δt.



Draw AN perpendicular to OB, and suppose the velocity AB resolved into the two, AN and BN; then, the former represents the resulting change of velocity in the normal direction, and the latter in the tangential.

The corresponding accelerations are represented by the limiting values of $\frac{AN}{\Delta t}$ and $\frac{BN}{\Delta t}$, respectively.

Again, let the angle BOA, or the angle between the tangents at a and b, when indefinitely small, be denoted by $d\phi$, and we have

$$AN = OAd\phi = vd\phi.$$

The normal acceleration is therefore

$$v\frac{d\phi}{dt} = v\frac{d\phi}{ds}\frac{ds}{dt} = v^2\frac{d\phi}{ds} = \frac{v^2}{\rho} \text{ (\textit{Diff. Calc.}, Art. 226)}, \quad (8)$$

where ρ represents the radius of curvature of the path at the point a.

Also in the limit we have $\frac{BN}{\Delta t} = \frac{dv}{dt}$. Hence the tangential acceleration is represented by $\frac{dv}{dt}$; as is also easily seen from equation (6).

In the case of uniform motion in a circle, since the velocity v is constant, the tangential acceleration vanishes, and the normal acceleration (which then becomes the total acceleration) is $\frac{v^2}{r}$, or $\frac{4\pi^2 r}{T^2}$, where r denotes the radius of the circle and T the time in which the circle is described.

The normal acceleration in this case is called the *centri-*

petal acceleration, as it is constantly directed towards the centre of the circle.

26. **Hodograph.***—In accordance with the method of the preceding Articles, if from any point O lines OA, OB, OC, &c., be drawn representing, in magnitude and direction, the velocities at the points a, b, c, &c., taken consecutively in the path of a particle, then the system of points A, B, C, &c., will lie on a new curve called the hodograph of the original curve, which is considered to be described by the point A as a moves along the given curve.

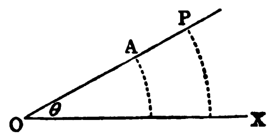
Since the lines AB, BC, &c., become ultimately tangents to the hodograph, it follows that the direction of the total acceleration at any point a is parallel to the tangent to the hodograph at the corresponding point A.

Also, since the total acceleration is measured by the limiting value of $\frac{AB}{\Delta t}$, it follows that the total *acceleration, at any point a, is represented by the velocity at the point A in the hodograph.*

We shall give some applications of this method subsequently, more especially in connexion with the treatment of Central Forces.

27. **Angular Velocity, Angular Acceleration.**—If the position of a point P moving in a plane be taken in polar coordinates, r and θ, with reference to a fixed origin O, then the rate of increase of the angle θ is called the angular velocity of P relative to the fixed point O. Hence, if ω denote the angnlar velocity at any instant,

we have $\omega = \frac{d\theta}{dt} = \dot{\theta}$.



Again, if P move along OA, its velocity is represented by $\frac{dr}{dt}$; if it move perpendicular to OA, its velocity is $r\,\frac{d\theta}{dt}$, or $r\omega$.

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* Sir W. R. Hamilton, to whom this method is due, employed this name (ὁδὸν γράφειν) (*Proceedings*, R. I. A., 1846, p. 344) in his discussion of the connexion between acceleration and motion. The hodograph is called the *curve of accelerations* by French writers on Mechanics.

Hence we easily see that the most general motion in the plane is one compounded of a *radial* velocity $\frac{dr}{dt}$, along with a perpendicular velocity $r\frac{d\theta}{dt}$.

If OP revolve uniformly, completing its revolution in T seconds, then its angular velocity, in circular measure, is obviously given by the equation

$$\omega = \frac{2\pi}{T}. \qquad (9)$$

Suppose OA taken equal to the unit of length, then the velocity of the point A, in its circular path, represents the angular velocity of the line OP.

Again, if the angular velocity of P be variable, its rate of increase is called its angular acceleration; hence the angular acceleration of P with regard to O is represented by $\frac{d\omega}{dt}$ or $\frac{d^2\theta}{dt^2}$.

If x and y be the coordinates of P, we have

$$x = r\cos\theta, \quad y = r\sin\theta;$$

consequently, when r is constant, we get

$$\left.\begin{aligned} \dot{x} &= -r\omega\sin\theta = -\omega y \\ \dot{y} &= r\omega\cos\theta = \omega x \end{aligned}\right\}. \qquad (10)$$

These give the components of velocity of any point which moves in a circle, in terms of the coordinates and the angular velocity.

28. **Accelerations along and perpendicular to the Radius Vector.**—Let x, y be the rectangular coordinates of the moving point P, and r, θ the corresponding polar coordinates, at the end of the time t; then $\ddot{x}$ and $\ddot{y}$ (Art. 23) represent the accelerations parallel to the axes; hence, by Art. 22, the acceleration, P, along the radius vector is

$$\ddot{x}\cos\theta + \ddot{y}\sin\theta, \text{ or } \frac{x\ddot{x} + y\ddot{y}}{r};$$

and the acceleration, T, perpendicular to the radius vector is

$$\ddot{y}\cos\theta - \ddot{x}\sin\theta = \frac{x\ddot{y} - y\ddot{x}}{r}.$$

To find expressions for these accelerations in terms of r and θ, we have

$$x\dot{x} + y\dot{y} = r\dot{r};$$

hence $$x\ddot{x} + y\ddot{y} + \dot{x}^2 + \dot{y}^2 = r\ddot{r} + \dot{r}^2;$$

but $$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2;$$

accordingly, $$x\ddot{x} + y\ddot{y} + r^2\dot{\theta}^2 = r\ddot{r};$$

therefore $$\frac{x\ddot{x} + y\ddot{y}}{r} = \ddot{r} - r\dot{\theta}^2.$$

Also $$x\ddot{y} - y\ddot{x} = \frac{d}{dt}(x\dot{y} - y\dot{x}) = \frac{d}{dt}(r^2\dot{\theta}).$$

Consequently, the acceleration along the radius vector is

$$P = \ddot{r} - r\dot{\theta}^2 = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2. \quad (11)$$

And that perpendicular to the radius vector is

$$T = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right). \quad (12)$$

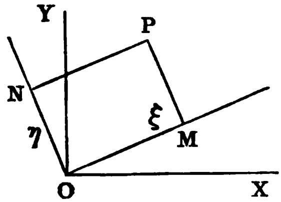
If the acceleration of the moving particle be always directed to the fixed point O, we have $T = 0$, and hence $r^2\dfrac{d\theta}{dt} = \text{constant}$; from which we infer that the radius vector describes equal areas in equal times round the point O.

Equations (11) and (12) above can otherwise be obtained with great facility by a method analogous to that employed in Art. 25.

29. **Areal Velocity, Areal Acceleration.**—It is obvious, geometrically, that $r^2d\theta$ represent double the area

described by the line OP in the time dt; consequently $\frac{r^2 d\theta}{dt}$ represents the rate of increase of double the area described by the point P round the point O. Hence $\frac{1}{2}\frac{r^2 d\theta}{dt}$ is called the *areal velocity* of the point P relative to the origin O. Similarly $\frac{1}{2}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right)$ represents the *areal acceleration* of P relative to the same origin.

30. **Moving Axes.**—In some cases it is necessary to refer the motion of a point in a plane to rectangular axes, which are themselves in motion. Thus let OX, OY be two fixed rectangular axes in the plane, and OM, ON be two moving axes.



Let P be any point in the plane; then $\xi = OM$, $\eta = ON$, where ξ and η are the coordinates of P, relative to the moving axes.

Also, if $\theta = \angle XOM$, we have $\omega = \frac{d\theta}{dt}$, the angular velocity of the moving axes. Then the motion of P is got by compounding the motions of M and N.

Now, by Art. 27, the components of the velocity of M are $\frac{d\xi}{dt}$ along OM, and $\omega\xi$ along MP. Likewise, the components for N are $\frac{d\eta}{dt}$ along ON, and $-\omega\eta$ along NP.

Hence, if u and v denote the components of the velocity of P, relative to the moving axes, we have

$$\left.\begin{aligned} u &= \frac{d\xi}{dt} - \eta\omega \\ v &= \frac{d\eta}{dt} + \xi\omega \end{aligned}\right\}. \qquad (13)$$

Again, by Art. 28, the acceleration of M along OM is

$\frac{d^2\xi}{dt^2} - \omega^2\xi$, and that along MP is $\frac{1}{\xi}\frac{d}{dt}(\omega\xi^2)$; with similar expressions for the accelerations of N.

Hence, finally, we get

$$\left.\begin{aligned} &\text{acceleration parallel to } OM = \frac{d^2\xi}{dt^2} - \omega^2\xi - \frac{1}{\eta}\frac{d}{dt}(\omega\eta^2); \\ &\text{acceleration parallel to } ON = \frac{d^2\eta}{dt^2} - \omega^2\eta + \frac{1}{\xi}\frac{d}{dt}(\omega\xi^2). \end{aligned}\right\} \quad (14)$$

31. **Units of Time and Space.**—With respect to the units of time and space, as well as of all other quantities, it should be remarked that the units assumed must in all cases be *finite magnitudes*. For instance, the unit of time may be taken as a second, an hour, a day, or any other finite interval of time, but it should never be assumed to be an *indefinitely small portion* of time; for if so, numbers which represent finite intervals of time become infinitely great, and accordingly arguments based on such an assumption become illusory and unmeaning when applied to finite intervals of time. This remark is requisite, as fallacious proofs are sometimes given in books on dynamics from overlooking this obvious principle.

The unit of time most universally adopted is a second, as already stated. Different units of length prevail in different countries. Since in this country the foot is the standard of length, and areas and volumes are each referred to units of their own, we shall sometimes employ such units for the purpose of illustrating mechanical principles by familiar examples. But, when desirable, we shall avail ourselves of the metric system. In it the unit of length is a metre (3·2809 feet, or 39·37079 inches). From this, by the simple processes of squaring and cubing, units of area and volume are derived; and decimal multiples and submultiples are respectively indicated by the use of Greek and Latin prefixes. For example, the centimetre is the hundredth part of the length of a metre. Again, one cubic decimetre is the measure of capacity called a litre, and is about 61 cubic inches, or 1·76 pints. We shall subsequently see that a cubic centimetre of distilled water at its greatest density

furnishes this system with another unit: to this the name gramme is applied. One thousand grammes are called a kilogramme, equivalent to about two and one-fifth pounds avoirdupois.

It should also be observed that in the numerical expression for an acceleration there is a double reference to the unit of time; so that, in strict accuracy, what we have called an acceleration of 7 feet per second should be called an acceleration of 7 feet per second per second. This mode of expression is, however, cumbrous, and quite unnecessary, since in ordinary language, as well as in mathematical deductions, it is assumed that velocities and their rates of change are referred to the same unit of time, unless the contrary be stated.

CHAPTER III.

LAWS OF MOTION.

SECTION I.—*Rectilinear Motion.*

32. **Motion in relation to Force.**—IN the preceding Chapters motion has been considered from a purely *kinematical* point of view; we now proceed to consider it in connexion with the force or forces by which it is produced.

The science of Rational Dynamics is usually founded on three principles, or Laws of Motion, which have been stated in their simplest form by Newton, and are fully verified by their agreement with experience. In the present Chapter it is proposed to discuss and illustrate various cases of application of these Laws, chiefly when the forces supposed to act are constant both in direction and magnitude. The discussion of motion produced by varying force will be dealt with subsequently. We follow Newton's method, commencing with the statement of his First Law.

33. **First Law of Motion.**—*A body continues in its state of rest, or of straight uniform motion, except in so far as it is compelled to alter that state by impressed force.*

This law asserts that a body has no power or tendency in itself to alter either its velocity or the direction of its motion: this is usually called the Law of Inertia of Matter.

Hence, if a body be conceived to be set in motion, and no external force act upon it afterwards, it should continue to move indefinitely in a right line with a uniform velocity.

Conversely, if a body be in a state of uniform rectilinear motion, we infer that the forces which act on it are in equilibrium. For example, if a train be in a state of uniform motion on a horizontal railway, we infer that the force arising from the action of the steam is exactly equal, and opposite to, the entire resistance arising from friction and resistance of the air.

Hence, all questions of *uniform rectilinear motion may be regarded as problems of equilibrium, and treated by the principles arrived at in Statics.* In all applications of the Laws of Motion to a body of finite dimensions, the only motion considered in this Chapter is one of pure translation.

Again, if the motion of a body be not uniform, or not rectilinear, we infer that it must be acted on by some external force or forces. The connexion between the motion produced and the force which produces it is contained under the next Law.

EXAMPLE.

A railway train is moving with constant velocity along a horizontal railroad. The resistance from friction, &c., for each carriage is one-hundredth part of the pressure. Find the tension of the couplings of the last carriage, if its weight be four tons.

In this case, since the motion is uniform, the tension of the couplings must be equal to the resistance to be overcome, or to the one-hundredth part of four tons, *i.e.* $89\frac{3}{5}$ lbs.

34. **Second Law of Motion.**—*Change of motion* is proportional to the impressed motive force, and takes place in the right line in which that force is impressed.*

As this statement is very comprehensive, it will be necessary to dwell on it with some detail, commencing with the case of a body under the influence of a force which acts uniformly and in the same right line during the motion. The body is supposed, in the first instance, to start from rest, and the *direction of the force to pass constantly through its centre of mass*, in which case the motion is one of translation† solely.

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* For the present we shall consider that it is one and the same body which is acted on by forces passing through its centre of mass, in which case the force varies directly as the velocity generated in the unit of time. We shall subsequently treat of the case where the mass acted on varies also. In that case, by the word "motus," here translated motion, we must understand *quantity of motion*.

† A force applied at the centre of mass of a rigid body is equivalent to an indefinite number of equal and parallel forces applied to the several equal particles of which the body is conceived to be constituted; but as the forces are equal, and the masses moved by each are equal, the velocities generated, in the same time, are also equal: hence the motion of the entire body is one of pure translation. The simplest case of this is that of bodies falling under the action of the force of gravity.

35. **Velocity Generated.**—Suppose a force to act uniformly on a body, and let f denote the velocity generated at the end of the first second (taken as the unit of time), then during the next second, in accordance with our law, the uniform force will generate an additional velocity of the same amount f; and in each successive second the force generates the same additional velocity; consequently the motion is in this case *uniformly accelerated*, and the velocity at the end of t seconds (Art. 17) is given by the equation

$$v = ft.$$

Again, if the body be supposed to start with the velocity v_0 in the direction in which the force acts, we shall have for the velocity v, at the end of the time t,

$$v = v_0 + ft, \tag{1}$$

as in Art. 17.

If the force act in a direction opposite to that of the motion it is called a *retarding force*; which, if uniform, will diminish the velocity by the quantity f during each second, and we shall have, as before, the equation

$$v = v_0 - ft.$$

The student should bear in mind that f in all cases is measured by the *velocity generated or destroyed in the movable in each second during the motion*; f consequently may always be regarded as an *acceleration*—a retardation being considered as a negative acceleration.

It may be observed that the entire reasoning in this Article depends on the following principle—contained in the Second Law of Motion—that the *change of velocity produced by a force in any time is independent of the previous velocity of the movable.*

The Second Law of Motion equally applies to the case of a body acted on by any number of forces, in which case it may be stated as follows:—

If any number of forces act simultaneously on a body, then, during any instant, each force produces the same change of motion in its own direction as if it had acted singly on the body.

From this it follows that forces are compounded in the

same manner as velocities. The law of the composition of forces was thus establishad by Newton—*Leges Motus*, Cor. 2.

36. **Space described in any Time.**—Since we have seen that in the case of a uniform force the velocity is uniformly accelerated or retarded, we can at once apply the results already arrived at in Arts. 18, 19.

Hence, the space described from rest, in the time t, is given by the formula

$$s = \tfrac{1}{2}ft^2. \qquad (2)$$

If the body start with an initial velocity v_0 along the line in which the force acts, we shall have

$$s = v_0 t \pm \tfrac{1}{2}ft^2, \qquad (3)$$

in which the upper or lower sign is taken according as the uniform force acts in the same or the opposite direction to that of the initial velocity.

It is plain that the space described in the first second from rest is $\frac{1}{2}f$, or *half the velocity acquired at the end of the second;* and, in general, the space described in any time from rest is half of that described by a body moving uniformly with the velocity acquired at the end of the time.

37. **Relation between Velocity and Space described.**—If the body start from rest, by eliminating t between the equations $v = ft$ and $s = \frac{1}{2}ft^2$, we get

$$v^2 = 2fs;$$

and, more generally, if v_0 be the initial velocity,

$$v^2 = v_0^2 \pm 2fs. \qquad (4)$$

From the preceding results it is seen that the question of rectilinear motion under the action of a constant force is completely solved whenever the value of the acceleration f can be determined. In a subsequent Article we shall show how this can be done in elementary cases, but before doing so we proceed to apply the preceding results to the important case of falling bodies.

38. **Vertical Motion.**—In order to get rid of the retardation caused by the resistance of the air, we shall sup-

pose the motion to take place in a vacuum. Under these circumstances it is found that all bodies, no matter what their density or chemical constitution may be, fall through the same vertical height and acquire the same velocity in the same time. That this is so is best established by means of pendulum experiments; but it can also be tested by allowing different bodies to fall in an exhausted receiver. We hence infer that the attractive force of the Earth acts equally on all bodies.

If g denote the acceleration due to the force of gravity, that is the *increment of velocity per second acquired by a body falling in a vacuum*, then, from what has been stated, the value of g is the same for all bodies at the same place on the Earth's surface.

Again, since at any place the force of gravity may be assumed as a constant force (*i. e.* within moderate distances from the Earth's surface), we may apply to the case of falling bodies the results arrived at in the preceding Articles by substituting g in place of f. Hence, if the body start from rest, we have

$$v = gt, \quad s = \tfrac{1}{2}gt^2, \quad v^2 = 2gs. \tag{5}$$

Again, if it start downwards with a given vertical velocity v_0,

$$v = v_0 + gt, \quad s = v_0t + \tfrac{1}{2}gt^2, \quad v^2 = v_0^2 + 2gs. \tag{6}$$

If the body be projected *vertically upwards* with a velocity v_0, gravity becomes a uniformly retarding force, and we have

$$v = v_0 - gt, \quad s = v_0t - \tfrac{1}{2}gt^2, \quad v^2 = v_0^2 - 2gs. \tag{7}$$

To find in this case the height H to which the body would ascend, we make $v = 0$ in the last equation, and we get

$$H = \frac{v_0^2}{2g}. \tag{8}$$

The time T of ascent is given in like manner by the equation

$$T = \frac{v_0}{g}. \tag{9}$$

The subsequent motion of the body is got from equations (5), in which we suppose the body to start from rest at the

height H. It immediately follows that the times of ascent and descent are equal, and that the body returns to its original position with the velocity with which it was projected upwards. For this reason we say that the *velocity* v_0 *is due to the height* H; and reciprocally, that the *height* H *is due to the velocity* v_0. We shall meet frequent applications of these expressions.

As the motion is supposed to take place in a vacuum, the preceding results can only be regarded as approximate for motion in the air.

39. **Variation of Gravity.**—It is found that the value of g varies, within small limits, from place to place on the Earth's surface. It increases with the latitude, and when referred to feet and seconds, has its least value, 32·091, at the equator, and its greatest, 32·255, at the pole. It also diminishes as the body is raised above the Earth's surface, since the attraction of the Earth varies as the inverse square of the distance from its centre. The value of g at London, referred to the same units, is 32·19, and this may be employed, in ordinary calculations, as an average value.

It will be seen subsequently that the rotation of the Earth on its axis has the effect of diminishing the velocity of a falling body; and, accordingly, the observed value of g is the difference between its value arising from the Earth's attraction and the component of the centrifugal acceleration in the vertical direction.

As a rough approximation we may assume $g = 32$; and, when numerical results are required, this may be taken as its value in these and all subsequent examples, unless otherwise specified, inasmuch as they are given chiefly for the purpose of familiarizing the student with the application of mechanical principles.

Examples.

1. Find the velocity acquired in 5 minutes by a falling body, assuming $g = 32·19$. *Ans.* 9657 feet.

2. In what time will a falling body acquire a velocity of 400 feet per second if it start from rest? *Ans.* 12·5 sec.

3. If a body move under the action of a constant force, its *average velocity* during any time is an arithmetical mean between its velocities at the commencement and the end of that time?

4. If one minute be taken as the unit of time, what should be taken as the value of g?

Ans. The velocity per *minute* acquired in one *minute* by a falling body, or 115,200 feet.

5. Two bodies start together from rest, and move in directions at right angles to each other. One moves uniformly with a velocity of 3 feet per second; the other moves under the action of a constant force: determine the acceleration due to this force if the bodies at the end of 4 seconds be 20 feet apart.

Ans. 2 feet per second.

6. If a uniform force generate in a body a velocity of 30 feet a second after describing 25 yards, find the acceleration. *Ans.* $f = 6$.

7. A stone is let fall from a height into a well, and is heard to strike the water after t seconds; find the depth of the well; assuming the velocity of sound to be V, and neglecting the resistance of the air.

The required height h is got by solving the equation

$$\frac{h}{V} + \sqrt{\frac{2h}{g}} = t.$$

In applying this equation practically, it may be observed that $\frac{h}{V}$ is, in all cases, small in comparison with t: accordingly, if we transpose and square, we get, neglecting $\frac{h^2}{V^2}$ in comparison with $\frac{2ht}{V}$,

$$h = \frac{gVt^2}{2(V + gt)}.$$

8. A person drops a stone into a well, and after three seconds hears it strike the water. If the velocity of sound be 1127 feet per second, find the depth of the water. *Ans.* 132·68 feet.

9. Prove that the spaces described by a falling body in successive equal intervals of time are proportional to the series of odd numbers.

10. A body moves from rest under the action of a constant force during four seconds, when the force is supposed to cease; in the next five seconds the body describes 200 feet; find the acceleration due to the constant force—(1) if one second; (2) if one minute be taken as the unit of time.

Ans. (1) 10; (2) 36000.

11. A body is projected upwards with any velocity, and t, t' denote the times in which it is respectively above and below the middle point of its path; find the value of $\frac{t}{t'}$. *Ans.* $\sqrt{2} + 1$.

12. Assuming g to be represented by 32 when the units of space and time are one foot and one second; what number would represent its value if one mile and one day be taken as the units? *Ans.* $45242181\frac{8}{11}$.

13. A ball is dropped from the masthead of a ship sailing n miles an hour. Through how many feet must it have fallen when the direction of its motion is inclined at 45° to the horizon? *Ans.* $\frac{121\,n^2}{3600}$.

40. **Acceleration Varies as Pressure.**—If we suppose different forces to act uniformly during equal times on the same body, it follows from the Second Law of Motion that the *forces will be to one another in the same ratio as the velocities generated in equal times.*

If we suppose the time of action to be one second, the velocities generated are represented by the corresponding accelerations f and f'. Also, if F, F' denote the statical* measures of the forces, *i. e.* the total pressures which they are capable of producing, we have

$$F : F' = f : f'. \tag{10}$$

If one of the constant forces be the attraction of the Earth, since its statical measure is W, or the weight of the body moved; and since g is the corresponding acceleration, we have

$$F : W = f : g; \tag{11}$$

hence

$$f = \frac{F}{W} g. \tag{12}$$

This equation enables us to determine the velocity generated in one second by a constant force at any place whenever the pressure F which measures the force is known, and also the weight of the body. We suppose, as stated already, that the body is rigid, and that the force F acts through its centre of mass. When f has been determined by the foregoing equation, and the force continues to act uniformly, we may apply the results arrived at in the preceding Articles to determine the subsequent motion (see Art. 37).

41. **Mass.**—Our ordinary experience suggests to us that the amount of the acceleration produced in a body by a force depends not only on the magnitude of the force but also on the body which is moved. When exact experiments are carried out it is found that the same force acting on different

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* The magnitude of a force is estimated in Statics by the weight which it is just capable of supporting. Thus, a force which is capable of supporting a weight of 112 lbs. is called a force of 112 lbs., &c.

bodies produces different accelerations, and that different forces acting on the same body produce accelerations proportional to the forces. Hence we conclude that the acceleration produced in the motion of a body by a force is equal to that force multiplied by a factor which is invariable for the same body, but which varies for different bodies.

Conversely, if F denote the magnitude of a force, and f that of the acceleration thereby produced, we have the equation

$$F = mf, \tag{13}$$

where m is always the same for the same body, but varies for different bodies. This quantity m is called the Mass of the body, and is estimated, like other quantities, by comparing it with a standard quantity of the same kind. It is found that at any fixed place on the Earth's surface the weight of a body (if permitted to accelerate its motion) produces an acceleration which is the same for all bodies (Art. 38). Now W being the weight and g the acceleration thereby produced, we have as above $W = mg$; but g is the same for all bodies at the same place, hence W is proportional to m; or, in other words, if there be two bodies whose weights are W, W', and whose masses are m, m', we have $\frac{W}{W'} = \frac{m}{m'}$. Hence, in order to find the ratio of the masses of two bodies, we have only to find the ratio of their weights at the same place.

Examples.

1. A uniform pressure of 6 lbs. is applied in a horizontal direction to a body of 10 lbs. mass placed on a smooth horizontal table. Find—(1) the velocity generated in one second; (2) that acquired after describing 500 yards along the plane. *Ans.* (1) $19\frac{1}{5}$; (2) 240.

2. If a uniform pressure of 3 lbs. produce a velocity of 10 feet in the first second, find the weight of the body acted on. *Ans.* 9·6 lbs.

3. Find the pressure which, acting uniformly during one second, will generate in a body of one ton mass a velocity of 10 miles per hour. *Ans.* 9 cwt. $18\frac{2}{3}$ lbs. pressure.

4. If a pressure of one ounce act uniformly on a body of one pound mass, find the velocity generated from rest in one minute. *Ans.* $\frac{15}{4}g$.

5. If a uniform force generate in a mass of 10 lbs. a velocity of 30 feet after describing 25 yards, find the statical measure of the force. *Ans.* $\dfrac{60}{g}$ lbs.

6. A pressure F, acting on a mass M, generates in it a velocity V, in the time T; find the value of F. *Ans.* $F = M\dfrac{V}{T}$.

7. A train of 100 tons acquires on a horizontal railroad in four minutes a velocity of 30 miles an hour; find the statical measure of the excess of the moving above the retarding pressure, each being assumed to be uniform.
Ans. 11 cwt. 1 qr. $23\frac{1}{3}$ lbs.

8. A train of 60 tons is impelled along a horizontal road by a constant pressure of 720 lbs. Supposing it to start from rest, find its velocity at the end of one minute—(1) neglecting friction; (2) assuming the resistance of friction, air, &c., to be 8 lbs. per ton. *Ans.* (1) $\frac{9}{28}g$; (2) $\frac{3}{28}g$.

9. If a uniform force of 6 lbs. produce in a second a velocity of 0·634 feet in a body, express the quantity of matter in the body in terms of cubic feet of water, assuming the weight of a cubic foot of water to be $62\frac{1}{2}$ lbs. and $g = 32{\cdot}19$. *Ans.* 4·87.

10. A mass of 450 lbs. is placed on a perfectly smooth table: a uniform horizontal pressure is exerted on it which increases its velocity 3 feet in every second; find the magnitude of the pressure in lbs.
Ans. 41 lbs., assuming $g = 32{\cdot}19$.

42. **Motion on a Smooth Inclined Plane.**—Let us suppose a body, *starting from rest*, to slide under the influence of gravity down a perfectly smooth inclined plane. Let i denote the inclination of the plane to the horizon, and W the weight of the body. Resolve W into its components, $W \sin i$ acting parallel to the plane, and $W \cos i$ perpendicular to the plane. The motion down the plane is evidently due to the former component, since the latter only causes pressure on the plane.

As the force along the plane is constant and acts in the direction of motion, we get, substituting $W \sin i$ for F in (12),

$$f = g \sin i. \tag{14}$$

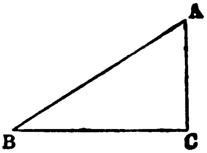
Hence, if $g \sin i$ be substituted for f in the formulæ in Arts. 36 and 37, we get

$$v = gt \sin i, \quad s = \tfrac{1}{2}gt^2 \sin i, \quad v^2 = 2gs \sin i. \tag{15}$$

We assume that the body slides without rolling along the

plane, as otherwise the motion would not be one of pure translation.

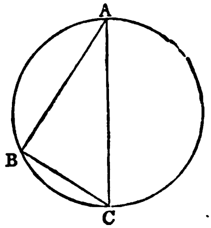
43. **Velocity acquired in Moving down an Inclined Plane.**—Let l represent AB, the length of the plane, and h its height AC; then if v be the velocity acquired on arriving at B, we have



$$v^2 = 2gl \sin i = 2gh. \qquad (16)$$

Accordingly, *the velocity acquired at any point in the descent of a body down a smooth inclined plane is that due to the vertical height through which the body has descended.* This is a particular case of an important principle which shall be subsequently considered.

44. **Time of Descending a Chord of a Vertical Circle.**—We next proceed to show that the time of descent down any chord of a vertical circle, starting from its highest point, is constant.



Let AC be the vertical diameter of the circle, AB any chord drawn from A. Join BC; then, $\sin i = \sin BCA = \frac{AB}{AC}$; and, if T be the time of descent for AB, we have, by (15),

$$AB = \tfrac{1}{2} g T^2 \frac{AB}{AC}, \quad \therefore T^2 = 2\frac{AC}{g};$$

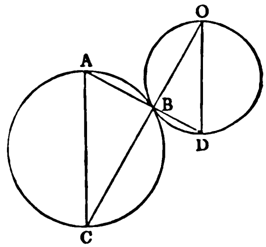
hence
$$T = 2\sqrt{\frac{a}{g}}, \qquad (17)$$

where a denotes the radius of the circle.

Hence, the time down any chord such as AB of this circle is constant. It can at once be seen, in like manner, that the time of descent down BC has the same value.

45. **Line of Quickest Descent to a Circle.**—To find the right line down which a particle under the action of gravity would descend in the shortest time from a given point O to a given vertical circle.

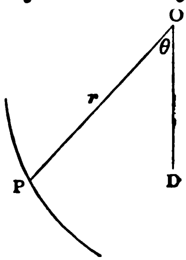
Draw AC, the vertical diameter of the circle, and join OC, meeting the circle in B, then OB is the line of quickest descent in question. For, join AB, and produce it to meet the vertical drawn through O in D. Then it is obvious that the circle described on OD as diameter touches the given circle in B; consequently the time of descent down OB being the same as that down any other chord of the circle OBD, drawn from O, is less than the time down any other right line drawn from O to meet the circle ABC.



The preceding method of investigation applies equally if the point O lie inside the given circle.

46. **Line of Quickest or Slowest Descent to any Curve.**—It is easily seen from the preceding Article that the determination of the right line of quickest or slowest descent to any given vertical curve from any point in its plane reduces to the problem of drawing a circle, touching the given curve and having the given point for its highest point.

The problem admits also of being treated by the ordinary method of maxima and minima, as follows:



Suppose the curve referred to polar coordinates, the given point O being taken as pole, and the vertical OD through it as prime vector; then, if t be the time of descent down any radius vector OP, we have

$$r = \tfrac{1}{2} g t^2 \cos\theta, \quad \text{or } t = \sqrt{\frac{2r}{g\cos\theta}}.$$

Accordingly, the time t is a maximum or a minimum when $\dfrac{r}{\cos\theta}$ is a maximum or a minimum.

To find the maximum or minimum values, assume $u = \dfrac{r}{\cos\theta}$; then since $\dfrac{du}{d\theta} = 0$, we have

$$\cos\theta \frac{dr}{d\theta} + r\sin\theta = 0. \tag{18}$$

The solutions are obtained by combining this equation with that of the curve.

To distinguish between the maximum and minimum solutions, we proceed to differentiate the equation

$$\frac{du}{d\theta} = \frac{\cos\theta \dfrac{dr}{d\theta} + r\sin\theta}{\cos^2\theta};$$

observing that, in this case, $\cos\theta \dfrac{dr}{d\theta} + r\sin\theta = 0$. Hence (*Diff. Calc.*, Art. 138), t is a minimum or a maximum according as $r + \dfrac{d^2r}{d\theta^2}$ is positive or negative.

These results can be readily verified from geometrical considerations.

Examples.

1. If the hypothenuse of a right-angled triangle be placed in a vertical position, prove that the times of descending from rest will be the same for each of its sides.

2. Prove that the velocity acquired down any chord, terminated at the lowest point of a vertical circle, is proportional to the length of the chord.

3. If the length of an inclined plane be 160 yards, and its inclination 30°, what velocity would a body acquire in descending it?

Ans. 40 yards per second.

4. A body slides down a smooth inclined plane of given height; prove that the time of descent varies as the length of the plane.

5. Find the inclination of a plane, of given length l, so that the velocity acquired in moving down it shall be of a given amount V. *Ans.* $\sin i = \dfrac{V^2}{2gl}$.

6. Given the base a of an inclined plane, find its height so that the horizontal velocity acquired by descending it may be the greatest possible.

Ans. $h = a$.

7. Find the gradient in a railway so that a carriage descending the plane by its own weight may move through one quarter of a mile in the first minute; and find how far the carriage will move in the next minute, friction being neglected.

(1) $\sin i = \frac{11}{480}$; (2) $\frac{3}{4}$ of a mile.

8. A body is attached by a string to a point in a smooth inclined plane, on which it rests: if it be projected from its position of rest up the plane with a velocity just sufficient to take it to the highest point to which the string allows it to go, find the time of its motion.

Ans. $t = 2\sqrt{\dfrac{l}{g\sin i}}$, the length of the string being l.

9. A groove is cut in an inclined plane, making an angle a with the intersection of the plane and the horizon. If a heavy particle be allowed to descend the groove (supposed smooth), prove that its acceleration is $g \sin i \sin a$; where i denotes the inclination of the plane.

10. If two vertical circles have a common highest point, then if any line be drawn from that point, the time of descending the portion intercepted between the circles is constant.

11. Find the right line of quickest descent from a point to a given right line lying in the same vertical plane as the point.

12. Find the right line of quickest descent from a given right line to a given vertical circle.

13. Find the lines of quickest and slowest descent between two vertical circles which lie in the same plane.

14. A parabola whose latus rectum is p is placed in a vertical plane, with its axis horizontal. Find the inclination of the normal terminated by the axis down which a particle would descend in the shortest time, and find the time of its descent.

Ans. $i = 45°$, time $= \sqrt{\dfrac{2p}{g}}$.

15. Find the latus rectum of a parabola, so that when it is placed in a vertical plane with its axis horizontal the least time in which a particle falls from rest down a normal from the curve to the axis may be one second.

16. Prove that the chords of quickest and slowest descent from the highest or to the lowest point of a vertical ellipse are at right angles to each other, and parallel to the axis of the curve.

17. Show immediately, from equation (18), that the right line of quickest descent from a given point to a given curve makes equal angles with the normal at its extremity and the vertical; and verify the result geometrically.

18. An ellipse is placed with its major axis vertical; find the semi-diameter along which a particle will descend in the shortest time possible from the circumference to the centre.

Ans. It makes with the axis major the angle $\sec^{-1}(e\sqrt{2})$, where e is the eccentricity. If $e < \dfrac{1}{\sqrt{2}}$, the line of quickest descent is the axis major.

19. An ellipse is placed with its major axis vertical; find the line of quickest descent from the upper focus to the curve.

Ans. It makes with the axis major the angle $\cos^{-1}\dfrac{1}{2e}$. If $e < \frac{1}{2}$, the axis major is the required line.

20. AB is a quadrant of a circle whose centre is O, the radius OB being horizontal; C is a point on the quadrant, and the angle $BOC = \theta$. Show that the time of falling from A to C is to that of falling from C to B as $\sqrt{\cos\dfrac{\theta}{2}}$ to $\sqrt{\sin\dfrac{\theta}{2}}$.

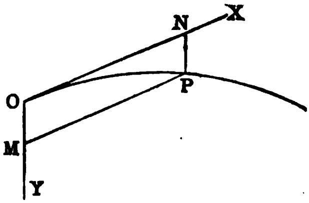
$\dfrac{l}{r} = 1 + e \cos. \theta.$

SECTION II.—*Parabolic Motion.*

47. **Path of a Projectile.**—We have hitherto considered rectilinear motion; we now proceed to the case of a body projected in any direction, and acted on only by the force of gravity, which is supposed to be uniform.

In this case it is easily shown that *the path* described by the projectile is a parabola.*

For, suppose a body projected from O with the velocity V, in the direction OX, and draw OY vertically downwards.



Let ON be the space which the body, moving with the velocity V, would describe in t seconds; then, if no force were to act on the body, N would represent its position at the end of that time.

Again, as the force of gravity acts in the direction OY, it will produce its effect in that direction, by the Second Law of Motion, independently of the previous velocity of the body: *i. e.* it will produce the same effect as if the body fell freely from rest. Measure off, accordingly, $OM = \frac{1}{2}gt^2$; then OM represents the space moved through in the vertical direction in the time t.

Complete the parallelogram $OMPN$, and by the combined effect of the two motions P will be the position of the projectile at the end of the time t.

Let x, y be the co-ordinates of P referred to the axes OX and OY, and we have

$$x = ON = Vt, \quad y = OM = \tfrac{1}{2}gt^2.$$

If t be eliminated between these equations, the equation of the path described is

$$x^2 = \frac{2V^2}{g}y. \qquad (1)$$

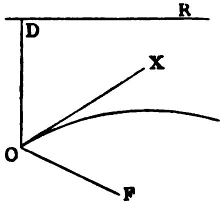
* As before, by the path described by a body we understand the path described by its *centre of mass.*

This equation represents a parabola, touching OX and having its axis vertical.

If H be the height due to the velocity V (Art. 38), the equation of the parabola becomes

$$x^2 = 4Hy. \qquad (2)$$

48. **Construction for Focus and Directrix.**—From the preceding equation it follows (Salmon's *Conic Sections*, Art. 214) that H is the distance of O from the focus of the parabola, and also from its directrix.



Hence, if OD be measured vertically upwards equal to H, and DR drawn in a horizontal direction, the line DR will be the directrix of the parabolic trajectory.

Also if OF be drawn through O, making the angle XOF equal to the angle XOD, and if we take $OF = OD$; then F will be the focus of the trajectory.

Hence, as the focus and directrix of the curve are known, it is completely determined.

Again, the *velocity at any point in the trajectory is equal to that which the body would acquire in falling from the directrix.*

We have seen that this property holds good for the point of projection: moreover, after passing through any point the body will move in the same path as if it had been projected from that point, in the direction and with the velocity that it has at the instant; therefore the property is true for any point in the path.

Hence, whenever the velocity at any point is given, the position of the directrix is completely determined.

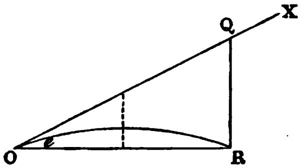
Definition.—The angle which the direction of projection makes with the horizontal line is called the *angle of elevation* of the projectile.

49. **Horizontal Range and Time of Flight.**—Let R be the point in which the projectile strikes the horizontal plane through O; then OR is called the horizontal range,

and the time T of describing the corresponding path is called the time of flight.

Through R draw RQ in the vertical direction.

Let $OR = R$, $\angle\, QOR = e$; then we have



$$OQ = VT, \quad QR = \tfrac{1}{2}gT^2.$$

But $QR = OQ \sin e$; hence we get

$$T = \frac{2V \sin e}{g}. \qquad (3)$$

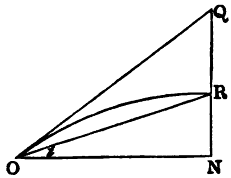
Also $R = OQ \cos e = VT \cos e = 2\dfrac{V^2}{g} \sin e \cos e$;

therefore $$R = 2H \sin 2e. \qquad (4)$$

If V be given, the horizontal range is the greatest when $\sin 2e = 1$, or $e = 45°$.

The maximum horizontal range is accordingly $2H$, or double the height due to the velocity of projection.

50. **Range and Time of Flight for an Oblique Plane.**—First suppose it an ascending plane, and let i be its inclination, and e the angle of elevation QON. Then, as before, we have



$$OQ = VT, \quad QR = \tfrac{1}{2}gT^2.$$

But in the triangle QOR, we have

$$\frac{QR}{OQ} = \frac{\sin(e-i)}{\cos i};$$

hence

$$\frac{\sin(e-i)}{\cos i} = \frac{gT}{2V},$$

or

$$T = \frac{2V}{g}\frac{\sin(e-i)}{\cos i}. \qquad (5)$$

Also the range

$$OR = OQ\frac{\cos e}{\cos i} = VT\frac{\cos e}{\cos i};$$

therefore

$$R = \frac{2V^2}{g}\,\frac{\sin(e-i)\cos e}{\cos^2 i}. \qquad (6)$$

In the case of a descending plane it is easily seen that the range and time of flight are obtained by changing the sign of i in the preceding results.

For given values of V and i, R becomes a maximum when $\sin(e-i)\cos e$ is a maximum, or when

$$\sin(2e-i) - \sin i \text{ is a maximum;}$$

but this is greatest when

$$2e - i = 90°, \quad \text{or} \quad e = \tfrac{1}{2}(90° + i).$$

Hence, the direction of elevation for a maximum range bisects the angle between the vertical and the inclined plane.

Again, since in this case $OR = RQ$, the maximum range and the corresponding time of flight are connected by the relation

$$R = \tfrac{1}{2}gT^2.$$

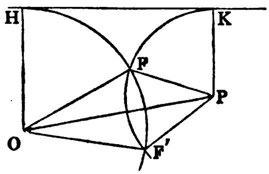
From the value of e found above, it follows immediately that the focus of the parabola, in this case, lies on the inclined plane.

51. *Given the velocity of projection to find the elevation in order to strike a given object.*—Here, in formula (6), we are given V, R, and i, to find e. Hence, $\sin(e-i)\cos e$ is given, and therefore $\sin(2e-i)$ is given, from which e can be determined.

The problem admits of a simple geometrical investigation also, as follows:—

Let O be the point of projection, and P the position of the given object. Then, since the velocity of projection is given, the position of the directrix HK is known.

Hence, with O and P as centres, describe circles touching the directrix, and let F, F' be their points of intersection. These points are obviously the foci of the two parabolic trajectories which satisfy the proposed conditions. Hence the problem admits in general of two solutions.

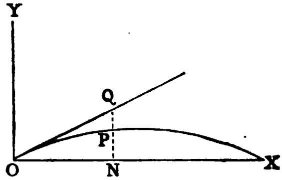


The corresponding directions of projection are found by bisecting the angles FOH and $F'OH$, as is obvious from the elementary properties of the parabola.

The problem becomes impossible when the circles do not intersect.

The range in the direction OP is obviously a maximum when the circles touch one another. In this case there is but one solution, and the focus of the parabola lies in the line OP, as already seen.

52. **Trajectory referred to Vertical and Horizontal Axes.**—Suppose OX and OY to be the horizontal and vertical lines drawn through the point of projection O, and let x, y be the coordinates of P, the position of the projectile at the end of any time t.



Let OQ be the direction of projection, and resolve the initial velocity V into its horizontal component, $V\cos e$, and its vertical, $V\sin e$. Then, since the force of gravity has no effect on the horizontal motion, the component $V\cos e$ remains constant during the motion; consequently we have

$$x = ON = Vt\cos e.$$

Also, for the motion in the vertical direction, we get (Art. 38),

$$y = Vt\sin e - \tfrac{1}{2}gt^2;$$

therefore

$$y = x\tan e - \frac{gx^2}{2V^2\cos^2 e}$$

$$= x\tan e - \frac{x^2}{4H\cos^2 e}. \qquad (7)$$

This equation represents a parabola, whose axis is vertical, as already seen.

Again, if v be the velocity at the point P, and ϕ the angle the direction of motion makes with the axis of x, we have

$$v \cos\phi = V \cos e, \quad \text{and} \quad v \sin\phi = V \sin e - gt;$$

hence
$$\begin{aligned} v^2 &= V^2 \cos^2 e + (V \sin e - gt)^2 \\ &= V^2 - 2g(Vt \sin e - \tfrac{1}{2}gt^2) \\ &= V^2 - 2gy = 2g(H - y). \end{aligned}$$

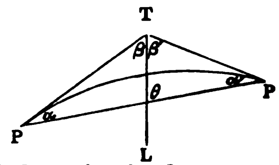
Hence, as already shown otherwise, the velocity at any point is that acquired by a body falling from the directrix.

53. **Height of Ascent.**—Since vertical and horizontal motions may be considered separately, it follows that the greatest height above the horizontal plane is that to which a body projected vertically with the velocity $V \sin e$ would ascend. This by (Art. 38), is

$$\frac{V^2 \sin^2 e}{2g}, \quad \text{or } H \sin^2 e.$$

Also, the time of ascent is $\dfrac{V \sin e}{g}$, from the same Article: a result which can also be obtained by finding the maximum value of y in equation (7). From these the same expressions as before for the range and the time of flight can be easily deduced: for, the whole time of flight is obviously double that of reaching the highest point; and the range is got by multiplying the value so found by $V \cos e$.

54. *If PT, $P'T$ be the tangents at two points P, P' on a parabolic trajectory, and v, v' the corresponding velocities, to prove that*



$$v : v' = PT : P'T. \qquad (8)$$

The line joining T to the middle point of PP' is vertical, being parallel to the axis of the parabola. Again, let

$$\alpha = \angle TPP', \quad \alpha' = \angle TP'P, \quad \beta = \angle PTL, \quad \beta' = \angle P'TL.$$

Then, since the horizontal component of the velocity at P is equal to that at P', we have

$$v \sin\beta = v' \sin\beta',$$

or

$$\frac{v}{v'} = \frac{\sin\beta'}{\sin\beta} = \frac{PT}{P'T}.$$

Also, since

$$\frac{PT}{P'T} = \frac{\sin\alpha'}{\sin\alpha},$$

we get

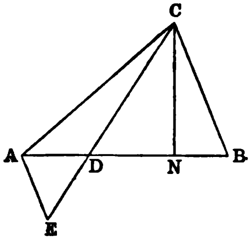
$$v \sin\alpha = v' \sin\alpha'. \tag{9}$$

55. **Lemma.**—If θ be the angle BDC which a right line CD drawn from the vertex makes with the base of a triangle ABC, we have

$$AB \cot\theta = BD \cot A - AD \cot B. \tag{10}$$

For, draw CN perpendicular to AB, and we have, by elementary geometry,

$$AB \,.\, DN = AN \,.\, DB - AD \,.\, BN.$$



Hence, dividing by CN,

$$AB \,.\, \frac{DN}{CN} = BD \,.\, \frac{AN}{CN} - AD \,.\, \frac{BN}{CN},$$

or

$$AB \,.\cot\theta = BD \cot A - AD \cot B.$$

Again, if α and β be the angles which CD makes with AC and BC respectively, we have

$$AB \cot\theta = AD \cot\alpha - BD \cot\beta.$$

This follows at once by drawing AE parallel to BC, and applying the preceding result.

56. *Being given the direction and the velocity of projection, to find the velocity with which a projectile would strike an oblique plane, and also the direction of its motion at the instant of impact.*

Let i be the inclination of the plane to the horizon; then, by the preceding lemma (see figure on last page),

$$\cot\alpha - \cot\alpha' = 2\tan i. \tag{11}$$

Hence, the angle α' is determined from the known angles α and i.

Again, since $v \sin a = v' \sin a'$, we have $v' = \dfrac{v \sin a}{\sin a'}$, which determines v'.

If the projectile impinge at right angles on the plane, we have $a' = 90°$; therefore $\cot a = 2 \tan i$, which determines a, or the corresponding angle of elevation. Also the velocity with which the projecticle strikes the plane is $v \sin a$ in this case.

57. **Motion on a Smooth Inclined Plane.**—In our discussion of motion on an inclined plane in Art. 42 the movable was supposed to start from rest: in this case the motion is rectilinear. It is also rectilinear if the initial motion has place in the direction of the line of greatest slope in the plane. But when the body is projected along the plane in any other direction the problem is the same as that previously discussed, namely, the motion of a projectile acted on by a constant force, parallel to a given direction. Its path along the plane is, accordingly, a parabola; and its axis is in the direction of the line of greatest slope.

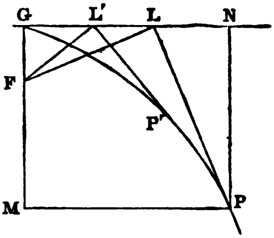
58. **Morin's Apparatus.**—We conclude with a short description of the apparatus, designed by Poncelet, and constructed by Morin, for experimentally exhibiting the laws of falling bodies.

A cylinder is made by clock-work mechanism to revolve around a fixed vertical axis. A weight is suspended at the summit of the cylinder close to the outer surface and between two vertical guides. When the rotation has become perfectly uniform, the weight is allowed to fall. A pencil, attached to the falling weight, is so arranged as to trace a line on a sheet of paper, which is wrapped tightly around the revolving cylinder. When the paper is taken off and unrolled on a plane surface, the curve traced on it by the pencil is found to be a parabola.

That this curve is a parabola, may be shown in the following manner:—

Let $GP'P$ represent the curve traced out by the pencil.

Draw the tangent GL to the curve at the initial point G, and at any point P draw the tangent PL, and erect LF perpendicular to it at the point L. Make a corresponding construction for the other points on the path; then the lines LF, $L'F$, &c., are all found to intersect in a common point F. This is a characteristic property of the parabola which has its focus at F, and its vertex at G. Having found the curve to be a parabola, we can show that the motion of the weight has been uniformly accelerated. Let PM, PN be the coordinates of P, referred to the axes GL, GF, then if t denote the time in which the moving weight arrived at the position P, the line PM will be equal to the arc of the circle through which a point on the circumference of the cylinder has rotated in the time t. Let V denote the constant velocity of any point on the circumference of the cylinder, and we get $PM = Vt$.



Again, from the property of the parabola,

$$PM^2 = 4FG \times MG.$$

Accordingly,

$$MG = \frac{PM^2}{4FG} = \frac{V^2}{4FG}t^2;$$

but MG is the space through which the weight has descended vertically in the time t; hence the spaces described by the falling body vary as the squares of the times; its motion consequently is uniformly accelerated.

Comparing with the equation $s = \frac{1}{2}gt^2$, we get $g = \frac{V^2}{2.FG}$; that is, the distance of the focus of the parabola from its summit is equal to the height due to the velocity of a point on the surface of the rotating cylinder.

The student can easily prove that the parabola described is the same as that of a body projected horizontally from a point with the velocity V.

59. In the preceding investigations we have neglected the effects of the resistance of the air. When this is taken into account the problem becomes one of great uncertainty, arising from the law of resistance of fluids not being accurately known, and from the difficulties still remaining in the integration of the equations of motion, when the law of resistance is assumed. The most generally received theory is that the resistance of fluids is proportional to the square of the relative velocity of the fluid and the movable. When the resistance of the air is taken into account, it is easily shown that the preceding results are not even approximate in cases of high velocity; such, for instance, as shot and shell projected by artillery.

Examples.

1. Determine the elevation of a projectile, so that its horizontal range may be equal to the space to be fallen through to acquire the velocity of projection.
Ans. $e = 15°$.

2. If a number of particles be projected simultaneously from the same point with a common velocity, but in different directions, prove that at any subsequent instant they will all be situated on the surface of a sphere.

3. Given the horizontal range and the time of flight of a projectile; find its initial velocity and angle of elevation.

4. If a body be projected obliquely on a smooth inclined plane, the path in which it moves will be a parabola. Find the position of the focus and directrix of the parabola when the initial velocity and direction of motion are given.

5. Given the velocity with which a shot is projected from a certain point; find the locus of the extremities of the maximum ranges on inclined planes passing through that point.

6. If a body be projected with a velocity of 100 feet per second from a height of 66 feet above the ground, in a direction making an angle of 30° with the horizon; find when and where it will strike the ground.
Ans. Time $= 4\frac{1}{8}$ sec. Range $= 357{\cdot}23$ feet.

7. If A, B be two points on a parabolic trajectory; prove that the time of passage from one to the other is proportional to $\tan\phi - \tan\phi'$; where ϕ, ϕ' represent the inclinations to the horizon of the tangents drawn at A and B.

8. Given the initial velocity, find the angle of elevation that a projectile should just clear a wall at a given distance from the point of projection. Find also the distance at which the body strikes the ground afterwards.

9. A piece of ordnance, under proof at Woolwich, at a distance of 50 yards from a wall 14 feet high, burst, and a fragment of it, originally in contact with the ground, after just grazing the wall, fell 6 feet beyond it on the opposite side. Find how high it rose in the air.

10. When the velocity of projection is given, all the parabolas which can be described in the same plane by a projectile are enveloped by a fixed parabola: prove this, and hence find the maximum range on a given plane.

11. A body is projected with a velocity of 100 feet, in a direction inclined at an angle of 60° to the horizon: find its least velocity during the motion, and the time of attaining it. *Ans.* 50 feet; 2·7 seconds.

12. If two bodies be projected simultaneously, with a common velocity, from the same point on an oblique plane, one upwards and the other downwards, and if the directions of their projection make equal angles with the inclined plane, show that the times of flight are equal. The motion is supposed to take place in a plane perpendicular to the inclined plane.

13. With what velocity should a projectile be discharged at an elevation of 30°, so as to strike an object at a distance of 2500 feet on an ascent of 1 in 40?

14. Find the latus rectum of the parabola described by a projectile.

The velocity of the *highest* point of the path is $V \cos e$, but it is also equal to the velocity acquired in falling from the directrix; therefore the latus rectum is $\frac{2V^2}{g} \cos^2 e$.

15. If a body be projected from the point A in the direction of AC, and from any point C in the line a vertical line CD be drawn, meeting the curve described by the projectile in D; again, if B, the middle point of AC, be joined to D, show that BD will be the direction of the motion at D, and that the velocity at D will be to that at A as BD is to AB.

16. A number of bodies slide from rest down the chords of a vertical circle, starting from its highest point, and afterwards move freely: prove that the locus of the foci of their paths is a circle whose radius is half that of the given circle.

17. If bodies be projected from the same point with velocities proportional to the sines of their elevations, find the locus of points arrived at in a given time.

18. Two bodies are projected simultaneously in different directions from the same point, with given velocities: prove that the line which connects their positions at each instant moves parallel to a given direction.

19. Two particles are projected from a point with equal velocities, their directions of projection being in the same vertical plane—t, t' being the times taken by the particles to reach their other common point, and T, T' the times of reaching their highest points. Show that $tT + t'T'$ is independent of the directions of projection.—*Camb. Trip.*, 1876.

20. If two particles be describing the same parabolic trajectory, prove that the right line connecting them envelopes an equal parabola.—*Ibid.*

21. A train is moving at the rate of 60 miles an hour when a ball is dropped from the roof inside one of the carriages. Prove that the ball describes a parabola in space, and find the position of the axis and directrix.

If the height of the carriage be 9 feet, and the ball rebound from the floor without loss of velocity, describe by means of a figure the path of the ball in space so long as the motion continues.

SECTION III.—*Friction.*

60. **Laws of Dynamical Friction.**—Before completing our discussion of motion under the action of a constant force, it is desirable to make a few observations on the resistance, arising from friction, which takes place when one body slides on another. We shall consider only the case of motion along a fixed plane, and shall assume that the roughness of the plane is the same throughout. Under these circumstances the laws of friction—as established by experiment—may be stated as follows :—

(1). The resistance caused by friction against the motion of a body sliding on a uniformly rough plane is proportional to the normal pressure which the body exerts against the plane.

(2). It is independent of the amount of surface in contact.

(3). It is independent of the velocity of motion.

(4). The ratio of the friction, during the motion, to the normal pressure is called the *coefficient* of Dynamical friction.

(5). The friction between two substances in motion is in general less than the friction in the state bordering on motion, or the Statical friction.

(6). The mutual friction varies with the nature of the surfaces in contact, and can be much diminished in amount by the use of unguents, as also by polishing the surfaces in contact.

The student will observe that the laws of Dynamical friction are in every respect similar to those of Statical friction (Minchin's *Statics*, Arts. 34–36).

For fuller information on the laws of Friction the student is referred to Jellett's *Theory of Friction.*

61. **Motion on a Rough Horizontal Plane.**—Let W be the weight of a body sliding on a uniformly rough horizontal plane, and μ the relative coefficient of friction; then,

since in this case the normal pressure is represented by W, the friction is μW; and since it acts as a retarding force we get by Art. 40,

$$f = \frac{\mu W}{W} g = \mu g. \qquad (1)$$

Accordingly, substituting $-\mu g$ for f in the equations of Arts. 35, 36, and 37, we get

$$\left.\begin{aligned} v &= V - \mu g t \\ v^2 &= V^2 - 2\mu g s \\ s &= Vt - \tfrac{1}{2}\mu g t^2 \end{aligned}\right\}. \qquad (2)$$

By means of these equations the motion is completely determined whenever μ, the coefficient of friction, and V, the initial velocity, are known.

To find when the body is brought to rest by the friction, we make $v = 0$ in the first of these equations, and the required number of seconds is $\frac{V}{\mu g}$. Again, the space moved over before the body is brought to rest is given by $2\mu g s = V^2$.

62. **Motion on a Rough Inclined Plane.**—Suppose a body of weight W to slide on a uniformly rough plane, of inclination i; then resolving W into its components, $W \cos i$ and $W \sin i$; the former, $W \cos i$, represents the pressure on the plane; and accordingly the friction is represented by $\mu W \cos i$; and since it acts against the motion, we have for the total force producing motion down the plane the expression $W \sin i - \mu W \cos i$. If this value be substituted for F in equation (12), Art. 40, we get

$$f = g(\sin i - \mu \cos i). \qquad (3)$$

If ϕ be the limiting angle of resistance for the plane, *i. e.* if $\mu = \tan \phi$, the preceding formula becomes

$$f = g \frac{\sin (i - \phi)}{\cos \phi},$$

for a body sliding down the plane.

The corresponding equations connecting velocity, time,

and space, are had by substituting this value for f in the formulae of Arts. 36 and 37.

If the body be projected up the plane, in a direction at right angles to the intersection of the plane with the horizon, the retarding force is represented by $W \sin i + \mu W \cos i$: hence the value of f becomes

$$f = -g(\sin i + \mu \cos i) = -g \frac{\sin(i+\phi)}{\cos \phi}, \qquad (4)$$

when we introduce for μ its value $\tan \phi$. The equations connecting s, v, t can be found immediately, as before.

Examples.

1. A body is projected with a velocity of 100 feet per second along a rough horizontal plane: find, assuming $\mu = \frac{1}{12}$, (1) the time in which it is brought to rest by friction; (2) the whole space passed over.

Ans. (1) $37\frac{1}{2}$ seconds; (2) 625 yards.

2. A body is projected with a velocity of 100 yards per minute along a rough horizontal plane, and is brought to rest in 10 seconds: find the coefficient of friction. *Ans.* $\mu = \frac{1}{64}$.

3. A train, of ten tons weight, is impelled by steam along a horizontal railroad with a constant pressure of 630 lbs. If the friction be 7 lbs. per ton, calculate—(1) the velocity, in miles per hour, after moving from rest for one minute; and (2) the space described in that time; neglecting the resistance of the air, &c.

Ans. (1) $32\frac{8}{11}$ miles; (2) 480 yards.

(*b*) If the steam be shut off, find how far the train would run before it is brought to rest by friction. 2 miles 320 yards.

4. A body projected with a velocity of 30 feet is brought to rest after sliding 100 yards on a rough horizontal plane; find the coefficient of friction.

Ans. $\frac{3}{64}$.

5. A body is projected up a plane, of 20 yards length and 30° inclination, with a velocity of 50 feet per second: find the coefficient of friction that the body should just arrive at the top of the plane.

Ans. $\mu = \dfrac{29}{96\sqrt{3}}$.

6. Two masses, M, M', connected by a string, slide down a rough inclined plane in a vertical plane at right angles to the intersection of the former with the horizon: if the coefficients of friction be μ and μ', respectively, prove that the acceleration down the plane is $g\left(\sin i - \dfrac{M\mu + M'\mu'}{M + M'} \cos i\right)$.

7. A body slides down a rough roof and afterwards falls to the ground: find the whole time of motion.

8. Several bodies start from the same point and slide down different inclined planes of the same roughness: find the locus of their positions after the lapse of a given time. Find also the locus of the positions arrived at with a common velocity.

9. A rough plane makes an angle of 45° with the horizon; a groove is cut in the plane making an angle α with the intersection of this plane and the horizontal plane; if a heavy particle be allowed to descend the groove from a given height h find the velocity with which it reaches the horizontal plane.

Ans. $\sqrt{\dfrac{2gh(\sin\alpha - \mu)}{\sin\alpha}}$.

10. A body moves from rest down an inclined plane whose inclination is 30°, and limiting angle of resistance 15°: find the velocity acquired if the length of the plane be 200 feet.

Here $v^2 = 400g \tan 15°$; therefore $v = 58{\cdot}56$ feet per second.

11. A railway train is moving up an incline of 1 in 120 with a uniform velocity. Find the tension of the couplings of the carriage which is attached to the engine; assuming the weight of the train (exclusive of the engine) to be 80 tons, and the friction 8 lbs. per ton.

Ans. 19 cwt. $5\frac{1}{3}$ lbs.

12. In the same case, if the acceleration of the train be 2 feet per second, find the tension of the couplings.

Here we must add to the preceding $W\dfrac{f}{g}$, *i.e.* 5 tons; and the entire tension is nearly 6 tons.

Section IV.—*Momentum.*

63. **Force measured by Quantity of Motion generated in Unit of Time.**—*The product of its mass and the velocity which a body has at any instant* is called its *quantity of motion* or *momentum* at that instant. Accordingly we conclude, from equation (13), Art. 41, that F varies as the *quantity of motion it can generate in one second* (taken as the unit of time), the force being supposed to act uniformly during that time.

Again, since the velocity (g) which gravity can produce in one second is the same for all bodies, the quantity of motion gravity can generate in one second in a falling body of mass m is represented by mg; hence, in this case, we have

$$W = mg;$$

in which the units of mass and weight are connected in such a manner that when one is fixed the other is also determined.

64. **Absolute Unit of Force.**—In accordance with equation (13), Art. 41, the unit of force is defined as the *force which, acting uniformly during the unit of time on a unit of mass, produces a unit of velocity.* This is called by Gauss the *absolute unit of force.*

The most convenient *unit of mass* in the British Isles is the mass contained in one *standard pound* avoirdupois.

Hence, adopting as before a second as the unit of time, and a foot as the unit of length, the absolute unit of force is that which, acting during one second, would produce in a standard pound mass a velocity of one foot per second. This unit of force is sometimes called a *poundal.* Hence, if $g = 32{\cdot}19$ with reference to the preceding units, the *unit of force* is $\frac{1}{32{\cdot}19}$ part of the attraction of the earth, at London, on a standard pound; *i. e.* about half an ounce, approximately.

In the metric system the force which in one second would generate a velocity of one centimetre per second in a gramme of matter is called a *dyne.* Hence, since 1 lb. = 453·6 grammes, and 1 foot = 30·48 centimetres, one poundal is approximately 13825 dynes.

65.* **Gravitation Units of Force and Mass.**—In practical questions concerning bodies on the earth's surface, it is in general more convenient to measure forces by weights, and to speak of a force of so many pounds weight. In this system the unit of force is the weight at some definite place (London) of the pound mass; or of a kilogramme when the metric system is taken. This is called the gravitation or statical measure of force; and since the unit of force in this system, acting on one pound mass for one second, produces a velocity of 32·19 feet per second, we see that this unit is 32·19 times the absolute unit. Moreover, since the weight of a body varies, within certain small limits from place to place (Art. 38), when scientific accuracy is required we must correct for the change in the value of g due to any difference in altitude or latitude from those of the place to which the standard was originally referred.

In practice this correction seldom requires to be taken into account, as the variation in the value of g is generally too small to affect the result appreciably (Art. 39).

EXAMPLES.

1. An ounce being taken as the unit of mass, a second as the unit of time, and an inch as the unit of length, compare the unit of force with the weight of one pound.

Here the unit of force is that which in one second would generate a velocity of one inch per second in an ounce mass; and therefore is $\frac{1}{12 \times 16 \times 32 \cdot 2}$ part of the weight of one pound, or 1·25 grains.

2. Determine the unit of time in order that g may be expressed by unity when the foot is the unit of length.

Ans. $\frac{1}{8}\sqrt{2}$ seconds.

3. Find the units of space and time in order that the acceleration of a body falling in vacuo, and the velocity it acquires in one minute, may respectively be the units of acceleration and of velocity.

66. **Two Classes of Forces.**—There are two classes of forces to be considered in Dynamics: one, such as gravity and those hitherto discussed, which require a finite time to produce a finite change of velocity. Forces of this class, when uniform, are, as has been stated, measured by the change produced in one second (taken as unit of time) in the momentum of the body acted on. There is another class, called ordinarily *impulses*, such as *blows*, sudden *impacts*. &c., which act only during a very *short* time, but are capable of producing a finite change of velocity in that time.

These are sometimes called *instantaneous* forces; it is necessary, however, to observe that force in all cases requires some time to produce its effects, though that time may be exceedingly small. In fact, we cannot conceive that a force could produce any change in the velocity of a body if its time of action were absolutely nothing.

Forces of the former class are frequently styled *finite* or continuous forces, to distinguish them from the other class, namely, *impulsive* forces.

It should be observed that whenever both *impulsive* and *finite* forces act at the same time on a body, the latter may in general be neglected in determining the motion at the instant; since the effects produced by them, in the time during which the impulsive forces act, are so small that they may be neglected in comparison with the effects of the impulses.

67. **Impulses.**—*The measure of an impulse,* i.e. of the entire action of a force of great intensity, which acts during a very short time, and then ceases, is the *whole change in the quantity of motion which it communicates to the body on which it acts.*

We may here observe that, if F be the instantaneous value of an impulsive force, and τ the time of action, the whole impulse is represented by $\int_0^\tau F dt$, in which, as already observed, τ is a very small interval of time.

68. **General Equations of Motion of a Particle.**—Suppose that the force F acts as before in the line of motion of the mass acted on, but that it varies continuously, then we may consider that in an indefinitely small portion of time its intensity is unaltered. The variable acceleration f, caused by it, is determined by the equation $F = mf$: hence, as in Art. 21, we have at any instant

$$F = mf = m\frac{dv}{dt} = m\frac{d^2s}{dt^2} = m\ddot{s}. \tag{3}$$

Hitherto the motion has been supposed rectilinear. In the case of curvilinear motion the last equation expresses the tangential component of the force, and it can be similarly seen (Art. 25) that the normal component is expressed by $m\frac{v^2}{\rho}$. We now proceed to consider the motion of a particle of mass m, under the action of any forces. If the particle be referred to a system of rectangular axes in space, and x, y, z, be the coordinates of its position at any instant, *i. e.* at the end of the time t, reckoned from any fixed instant, the components of its velocity parallel to the axes of coordinates are (Art. 12) represented by $\dot{x}$, $\dot{y}$, $\dot{z}$.

Resolve the whole force acting on the particle at the instant into three components, parallel to the axes of x, y, z, respectively; and let these components be represented by X, Y, Z; then, since by the Second Law of Motion each of these forces produces its change of velocity in its own

direction, we deduce from what precedes (see Art. 24) the equations

$$\left.\begin{aligned} X &= m\frac{d}{dt}\left(\frac{dx}{dt}\right) = m\frac{d^2x}{dt^2} = m\ddot{x}, \\ Y &= m\frac{d^2y}{dt^2} = m\ddot{y}, \quad Z = m\frac{d^2z}{dt^2} = m\ddot{z}. \end{aligned}\right\} \qquad (4)$$

These are called the differential equations of motion of the particle; and the solution of the problem depends in each case on the integration of these equations.

As already stated, the preceding equations hold for the motion of any rigid body, provided the direction of the force which acts on it always passes through its centre of mass.

69. In some problems the mass acted on constantly varies during the motion; in this case equation (3) becomes

$$F = \frac{d}{dt}(mv). \qquad (5)$$

For instance, suppose a ball projected vertically upwards, a chain of indefinite length being attached to it, and drawn up gradually by it; to investigate the motion.

Here, if m be the mass of the ball, μ that of a unit of length of the chain, and s the length of chain in motion at any instant, we have $M = m + \mu s$; and if $m = k\mu$, our equation gives

$$\frac{d}{dt}\left\{(k+s)\frac{ds}{dt}\right\} = -(k+s)g,$$

or

$$\frac{1}{2}\frac{d}{dt}\left[(k+s)^2\left(\frac{ds}{dt}\right)^2\right] = -(k+s)^2 g\frac{ds}{dt}.$$

Hence

$$(k+s)^2\left(\frac{ds}{dt}\right)^2 = C - \tfrac{2}{3}g(k+s)^3.$$

If V be the initial velocity, we have

$$k^2V^2 = C - \tfrac{2}{3}gk^3.$$

Hence

$$(k+s)^2\left(\frac{ds}{dt}\right)^2 = k^2 V^2 - \tfrac{2}{3} g \left((k+s)^3 - k^3\right). \qquad (6)$$

This determines the velocity at any height; also H, the height of ascent, is given by the equation

$$H = k^3\sqrt{1 + \frac{3V^2}{2kg}} - k. \qquad (7)$$

If $k = \infty$, this is easily seen to become $\frac{V^2}{2g}$, which agrees with Art. 38.

Section V.—*Action and Reaction.*

70. **Third Law of Motion.**—*Reaction is always equal and opposite to action: that is, the mutual actions of two bodies are always equal and take place in opposite directions.*

On this law Newton remarks as follows:—"If any person press a stone with his finger, his finger is pressed by the stone. If a horse draw a body by means of a rope, the horse also is drawn (so to speak) towards the body; for the rope being strained equally in both directions, draws the horse towards the body as well as the body towards the horse, and impedes the progress of one as much as it promotes that of the other. Again, if any body impinge on another, whatever quantity of motion it communicates to that other it loses itself (on account of the equality of the mutual pressure)."

Newton verified this law experimentally in the case of the collision of spherical bodies.—See *Scholium*, *Axiomata*.

He also showed that the law holds good in the case of the attraction of bodies, as follows:—

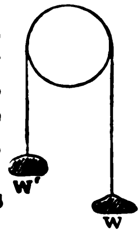
Let A and B be two mutually attracting bodies, and conceive some obstacle interposed by which their approach to one another is prevented. If the body A be acted on towards B by a greater force than B is acted on towards A, then the obstacle will be more urged by the pressure of A than by the pressure of B. The stronger pressure should prevail, and cause the system consisting of the two bodies and the obstacle to move in directum towards B; also, as the force is uniform the motion would be accelerated ad infinitum, which is absurd, and contrary to the first law of motion; for, by that law, such

a system, as it is not acted on by any *external* force, should continue in a state of rest or of uniform rectilinear motion.

71. **Stress, Forces of Inertia.**—The fact is that force is always exhibited as a mutual action between two bodies; and this phenomenon, regarded as a whole, is described by the term *stress*, of which action and reaction are but different aspects. Thus to the action of a force producing an acceleration of motion in a body corresponds an equal and opposite reaction against acceleration; this is called the *force of inertia* of the body. It thus follows that the force of inertia of any material particle must be equal and opposite to the resultant of all the forces which act on the particle, whether arising from the action of the other parts of the system or from that of forces external to the system. Hence, in the motion of any material system, since the actions and reactions of its different parts equilibrate in pairs, we infer that there is equilibrium between the external forces which act on the system and the several forces of inertia of the different particles of which the system is composed. This is equivalent to the celebrated principle introduced by D'Alembert, and called by his name, but which is directly implied in Newton's Scholium on the Third Law of Motion. This has been observed by many writers on Mechanics, but the connexion of Newton's Scholium with the modern theory of work and energy was first pointed out by Thompson and Tait: see their Treatise on Natural Philosophy, vol. i., pp. 247–8.

72. The laws of Motion, like every law of nature, must ultimately depend for their establishment on their agreement with experiment and observation. Accounts of the different apparatus that have been devised for the purpose of verifying these laws will be found in the books especially devoted to the purpose, such as Ball's Experimental Mechanics. The most complete proof of the laws of motion, however, is derived from Physical Astronomy. The Lunar motions, for instance, have been calculated from equations depending solely on these laws; and the observed and calculated positions are found to agree with a precision that could only arise from the perfect accuracy of the principles from which they were deduced.

One of the simplest contrivances for illustrating the laws of motion, in the case of falling bodies, is that devised by Atwood, which we shall now proceed to consider.



73. **Atwood's Machine.**—In its simplest form this machine may be regarded as consisting of two masses connected by a string which passes over a small fixed pulley. We shall neglect the weight of the pulley, and also that of the string, as well as the friction at the axle of the pulley.

Suppose W and W' to represent the weights of the bodies, of which W is the greater.

Let T denote the tension of the string at any instant: then considering the pulley as perfectly smooth, this tension, by the law of action and reaction, must act equally, and in opposite directions, on the two masses.

Accordingly, we may regard the body W as acted on by the pressure W downwards, and the tension T upwards; *i. e.* by the single force $W - T$ acting downwards—then, the corresponding acceleration f, from Art. 40, is given by the equation

$$f = \frac{W - T}{W} g.$$

Similarly, the upward acceleration of the other body is represented by $\dfrac{T - W'}{W'} g$.

Again, as the string is supposed *inextensible*, the velocities of the bodies at any instant are equal and opposite, and hence their accelerations also.

Accordingly we have

$$\frac{W - T}{W} = \frac{T - W'}{W'},$$

or

$$T = \frac{2WW'}{W + W'}. \qquad (1)$$

This determines the tension of the string. Again, we have

$$W - T = W\frac{f}{g}, \quad T - W' = W'\frac{f}{g};$$

therefore
$$W - W' = (W + W')\frac{f}{g},$$

or
$$f = \frac{W - W'}{W + W'}g. \qquad (2)$$

This determines the acceleration. By aid of it the velocity and the space described in any time can be readily deduced.

The most important advantage of this apparatus is that, by taking bodies of nearly equal weights, we can make the acceleration $\frac{W - W'}{W + W'}g$ as small as we please.

A complete account of Atwood's apparatus is beyond the scope of this treatise. In a subsequent place we shall consider the modification required when allowance is made for the mass of the pulley.

Examples.

1. A mass of 488 grammes is fastened to one end of a chord which passes over a smooth pulley. What mass must be attached to the other end in order that the 488 grammes may rise through a height of 200 centimetres in 10 seconds, assuming $g = 980$ centimetres? *Ans.* 492 grammes.

2. Two weights of 14 and 18 ozs. are suspended by a fine thread which passes over a smooth pulley, if the system be free to move; find how far the heavier weight will descend in the first three seconds of its motion, and also the tension of the string. *Ans.* 18 feet; and $15\frac{3}{4}$ ozs.

74. Suppose that one of the bodies is placed on a *smooth* horizontal table, and that the string, by which the bodies are attached, passes over a smooth pulley placed at the edge of the table; then, denoting the tension of the string by T, we have, as before,
$$f = \frac{W - T}{W}g.$$

Again, since the motion of the body on the smooth table arises from the tension T, we have
$$f = \frac{T}{W'}g.$$

Eliminating T, we get $f = \frac{W}{W+W'} g$. (3)

Again, equating the two values of f,

$$T = \frac{WW'}{W+W'}. \quad (4)$$

It may be observed that the tension of the string in this case is half of that in Atwood's machine for the same masses.

75. **Masses on Two Smooth Inclined Planes.**—Suppose two bodies, of weights W and W', placed on two planes, of inclinations i and i' to the horizon; and suppose the connecting string to lie in a vertical plane at right angles to the line of intersection of the two inclined planes, and to pass over a small pulley placed at the common summit of the planes; then, representing as before the tension of the string by T, since $W \sin i$ is the component of W acting parallel to the plane, we have

$$W \sin i - T = \frac{W}{g} f,$$

and

$$T - W' \sin i' = \frac{W'}{g} f.$$

Hence

$$f = \frac{W \sin i - W' \sin i'}{W+W'} g. \quad (5)$$

Also

$$T = \frac{WW'}{W+W'} (\sin i + \sin i'). \quad (6)$$

It is evident that W or W' will descend according as $W \sin i$ or $W' \sin i'$ has the greater value.

The results of the two former Articles are particular cases of the preceding; and are, accordingly, cases of the formulæ (5) and (6). We shall next consider the preceding problems for rough planes.

76. **Motion on Uniformly Rough Planes.**—Suppose two bodies connected as in Art. 74, and let μ denote the coefficient of friction for the horizontal plane.

The friction acting against the motion of W' is represented

by $\mu W'$; hence the pressure producing motion is $T - \mu W'$. We accordingly have the equation

$$T - \mu W' = \frac{W'}{g} f;$$

and also $$W - T = \frac{W}{g} f, \text{ as before.}$$

Hence we get $$f = \frac{W - \mu W'}{W + W'} g, \tag{7}$$

and $$T = \frac{WW'}{W + W'} (1 + \mu). \tag{8}$$

There can be no motion unless W is greater than $\mu W'$; as is also evident from elementary considerations.

Equation (7) may also be written in the form

$$\mu = \frac{W}{W'} - \frac{f}{g}\left(1 + \frac{W}{W'}\right); \tag{9}$$

from which μ can be determined when W and W' are known, f having been obtained by observation.

By this means the value of μ, the coefficient of dynamical friction was obtained for several substances by Coulomb.

Again, let μ, μ' be the coefficients of dynamical friction for the inclined planes, in Art. 75.

Since the pressures on the planes are represented by $W \cos i$ and $W' \cos i'$, respectively, the corresponding frictions are $\mu W \cos i$ and $\mu' W' \cos i'$; consequently the total pressure acting on W, down the plane, is represented by

$$W(\sin i - \mu \cos i) - T;$$

and we get $$W(\sin i - \mu \cos i) - T = \frac{W}{g} f.$$

And, similarly, $$T - W'(\sin i' + \mu' \cos i') = \frac{W'}{g} f.$$

Hence we have

$$f = \frac{W(\sin i - \mu \cos i) - W'(\sin i' + \mu' \cos i')}{W + W'} g, \tag{10}$$

and $$T = \frac{WW'}{W + W'} \{\sin i + \sin i' + \mu' \cos i' - \mu \cos i\}. \tag{11}$$

EXAMPLES.

1. If the two equal masses in Atwood's machine be each 1 lb.; required the additional mass which, added to one of them, would generate a velocity of one foot in each mass at the end of the first second. *Ans.* $\frac{2}{g-1}$ lbs.

2. In the same case find the tension of the string which connects the two masses. *Ans.* $\frac{g+1}{g}$ lbs.

3. Two smooth inclined planes are placed back to back: the inclination of one is 1 in 7, and of the other 1 in 10; a mass of 20 lbs. is placed on the first, and is connected by a string with a mass of 30 lbs. placed on the second plane. Find the acceleration of the descent, and the tension of the string.

Ans. $f = \frac{g}{350}$, $T = 2\frac{32}{35}$ lbs.

4. A mass of 10 lbs., falling vertically, draws a mass of 15 lbs. up a smooth plane, of 30° inclination, by a string passing over a pulley at the top of the plane. Find the acceleration, the space fallen through in 10 seconds, and the tension of the string. *Ans.* $f = \frac{g}{10}$; $s = 5g$; $T = 9$ lbs.

5. A mass, descending vertically, draws an equal mass 25 feet in $2\frac{1}{2}$ seconds up a smooth plane, inclined 30° to the horizon, by means of a string passing over a pulley at the top of the plane. Determine the corresponding value of g.

Ans. 32.

6. Given the height, h, of a smooth inclined plane, find its length so that a given weight P, descending vertically, shall draw another given weight Q up the plane in the least possible time. *Ans.* $\frac{2Qh}{P}$.

7. A mass P, falling vertically, draws another, Q, by a string passing over a fixed pulley: if, at the end of t seconds, the connecting string be cut, find the height to which Q will ascend afterwards. *Ans.* $\left(\frac{P-Q}{P+Q}\right)^2 \frac{gt^2}{2}$.

8. A mass, hanging vertically, draws an equal mass along a rough horizontal plane. If at the end of one second the string be cut, find how far the mass will move along the plane before it is brought to rest by the friction.

Ans. $\frac{(1-\mu)^2 g}{8\mu}$.

9. In what time will a mass of 2 lbs., hanging vertically, draw a mass of 30 lbs. along a smooth horizontal table of 36 feet length? *Ans.* 6 seconds.

10. If the plane in the last example be rough, and the coefficient of friction be $\frac{1}{25}$, find the time occupied. *Ans.* $3\sqrt{10}$ seconds.

11. In the previous example find at what instant during the motion the

string should be cut in order that the mass should just reach the edge of the table; and find the whole time of motion.

Ans. $12\sqrt{\frac{5}{13}}$ seconds, $\frac{3}{2}\sqrt{65}$ seconds (*q. p.*)

12. Two masses move on two smooth inclined planes, whose directions are at right angles to each other, and are connected by a string passing over the intersection of the planes. If the tension of the string be a maximum, find the inclination of either plane to the horizon. *Ans.* 45°.

13. In a single movable pulley, when there is equilibrium, the power and the weight hang by parallel strings. The weight being doubled, and the power halved, motion ensues. Prove that, if the friction and inertia of the pulley be neglected, the tension of the string will be unaltered. (*Camb. Trip.*, 1874.)

14. In general, if P be the weight attached to the movable pulley, and Q that to the other end of the string, prove that the tension of the string during the motion is $\frac{3PQ}{P+4Q}$; and that the acceleration of the movable pulley is $\frac{2Q-P}{P+4Q}g$; the friction and inertia being neglected as before.

Let T denote the tension of the string, f the acceleration of P, and f' that of Q; and we have $2T-P=P\frac{f}{g}$, $Q-T=Q\frac{f'}{g}$; but $f'=2f$; therefore, &c.

15. A train is travelling at a uniform rate on level rails. W is the weight of the fore portion of the train, and W' that of the brake-van at the end of the train. If the brakes be applied to the brake-van find the stress produced on the couplings between it and the next carriage, assuming μ to represent the coefficient of friction. *Ans.* $\mu\frac{WW'}{W+W'}$.

16. In Atwood's machine if the descending weight be a rigid homogeneous vertical rod AB, prove that the longitudinal stress at any point P of the rod, during the motion, is represented by $\frac{BP}{AB}T$, where T is the tension of the string.

17. In Atwood's machine if the pulley be rough, and if the effect of friction be to prevent motion until the tension of the string at one end be greater than that at the other by $\frac{1}{n}$th of the latter tension, prove that the effect on the acceleration will be the same as if the pulley remained smooth and the smaller weight were increased by $\frac{1}{n}$th.

18. In Atwood's machine, a mass P is attached to one end of the string, and two masses, Q and R, to the other end, where $Q+R>P$, and $P>Q$. After the united masses Q and R have descended s feet from rest, R is detached: find how much further Q will move before being brought to rest.

Let f be the acceleration in the first stage of the motion, f' that in the second, v the velocity at the instant R is detached, x the required distance; then

$$2fs=v^2=2f'x;$$

therefore
$$x=\frac{f}{f'}s=\frac{P+Q-R}{P+Q+R}\cdot\frac{P+Q}{P-Q}s.$$

CHAPTER IV.

IMPACT AND COLLISION.

77. **Collision of Homogeneous Spheres.**—In this chapter it is proposed to consider some elementary cases of impact of solids, but principally the collision of homogeneous spherical bodies, moving without rotation, whose centres, at the instant of collision, move in right lines lying in the same plane (all friction being neglected).

There are two cases to be considered, according as the centres of the spheres move in the same or in different right lines. The former is called direct, the latter oblique collision.

We commence with the former case, and at first suppose the centres to move in the same direction along the line.

78. **Direct Collision.**—Let M and M' represent the masses of the bodies, V and V' their velocities before, v and v' those after, collision. We also suppose M to impinge on M'.

The whole impact may be divided into two stages. During the first, the bodies compress each other, and the impinging body M, moving with a greater velocity than the other, accelerates its motion, until the exact instant at which their mutual compression is the greatest, when they are moving with a common velocity. During the second stage, the bodies tend to revert to their original shape, and the forces thus brought into play, called the *forces of restitution*, tend to cause the bodies to separate by still further diminishing the velocity of the impinging body, and increasing that of the other.

Suppose u represents the common velocity at the instant of greatest compression; then the quantity of motion lost by M during the first stage of the shock is $M(V-u)$, and that gained by M' is $M'(u-V')$.

These are the measures, by Art. 68, of the entire actions of the mutual forces during this stage of the collision; and, since by the Third Law of Motion, the forces must be equal and opposite, so also are their actions in the same time.

Hence, we have

$$M(V-u) = M'(u-V'),$$

or

$$u = \frac{MV+M'V'}{M+M'}. \tag{1}$$

In the case of perfectly *non-elastic* bodies, in which no forces of restitution are brought into play, the bodies would proceed after collision to move with this common velocity.

There is probably no case in nature of a perfectly non-elastic solid. All solid bodies with which we are acquainted have a tendency to recover, in different degrees, their original forms after being compressed. This tendency arises from their *elasticity*.

Bodies are said to be *perfectly elastic* when the forces of restitution, brought into play during the second stage, are exactly equal to the forces of compression, which act during the first. In this case the impinging body M would lose a further quantity of motion, $M(V-u)$, equal to that which it lost in the first stage. Therefore its velocity v, at the end of the shock, will be equal to $2u - V$.

In like manner we have $v' = 2u - V'$. Thus, in direct collision, we are enabled to determine the velocities, v, v', in the case of perfectly elastic bodies.

Bodies are, however, in general imperfectly elastic; that is, the whole force of restitution is less than that of compression. The Law of restitution, as derived from experiment, is usually stated thus:—*The ratio which the whole impulse of restitution bears to that of compression is constant while the impinging substances remain the same.* This ratio is usually represented by the letter e, having been by many writers called the modulus or *coefficient of elasticity;* but as this title is now employed in a different sense, we shall follow the current usage in adopting the name, *coefficient of restitution.*

From this law it follows that the quantity of motion lost by M during the second stage of the impact bears a constant

ratio to that lost during the first; and similarly for the quantity of motion gained by M'.

Accordingly

$$M(u-v) = eM(V-u), \quad M'(v'-u) = eM'(u-V').$$

Hence we get

$$v' - v = e(V - V'), \tag{2}$$

and

$$MV + M'V' = Mv + M'v'. \tag{3}$$

These equations enable us to determine the velocities, v, v', after impact, when those before impact are given, as also the masses M, M', and the coefficient of restitution.

It should be observed that equation (3) expresses that the total quantity of motion of the two bodies is the same after impact as before. This result is a particular case of a general principle which shall be subsequently considered (*see* Art. 83).

The result contained in (2) may be stated thus: *In direct collision* of two spheres the relative velocity after collision bears a constant ratio to the relative velocity before collision.*

This law was established by Newton, as the result of experiment (see his *Leges Motus*, scholium); and the coefficients of restitution for several substances, such as glass, ivory, steel, &c., were determined by him.

In more recent times a number of careful experiments were undertaken by Hodgkinson on the laws of restitution. The results are to be found in the Report of the British Association, 1834, and also in the Transactions of the Royal Society. His conclusions agree in the main with the law laid down by Newton, given above.

Some of the more important results of Hodgkinson's experiments may be briefly stated as follows:—

The coefficient of restitution diminishes slowly as the

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* This result is by some writers taken as the basis of the rational theory of collision. However, the method here given is that more usually adopted; it has the great advantage of connecting the problem directly with the consideration of force, and of illustrating the principle (Art. 67) that impulsive forces must be regarded as forces of great intensity whose time of action is very short.

velocity of impact increases; it is independent of the relative magnitude of the masses. In impact between bodies differing much in *hardness*, the coefficient of restitution is nearly equal to that between two specimens of the softer body.

No perfectly elastic body exists in nature: glass, however, may be regarded as nearly so, its coefficient being $\frac{15}{16}$, approximately, as determined by Newton.

When the mass M' is at rest, and very great in comparison with M, v' is very small, and we have approximately $v = -eV$.

Hence, if a body impinge perpendicularly, with a velocity V, upon a fixed plane, it will return back in its former direction of motion with a velocity represented by eV, where e is the coefficient of restitution.

79. **Height of Rebound.**—If a body fall from a height h on a fixed horizontal plane, then V, the velocity with which it strikes the plane, is equal to $\sqrt{2gh}$. The velocity of rebound is eV, or $e\sqrt{2gh}$: hence, if h' be the height to which it rebounds, we have

$$\sqrt{2gh'} = e\sqrt{2gh},$$

or

$$h' = e^2h. \tag{4}$$

In all cases of collision the student should be careful to give the proper algebraic signs to the velocities. The velocity V of the mass M is usually taken as positive; and hence the other velocities will have positive or negative signs according as they take place in the same or the opposite direction to that of V.

Examples.

1. If a mass M impinge directly on another mass M', at rest, find the relation between them when the impinging mass is reduced to rest by the collision.
Ans. $M = eM'$.

2. A ball of 6 lbs. mass, moving at the rate of 10 miles an hour, overtakes another of 4 lbs. mass, moving at 5 miles an hour: determine their velocities after collision, assuming $e = \frac{1}{2}$; the impact being supposed direct.
Ans. $v = 7$; $v' = 9\frac{1}{2}$.

3. Find the corresponding velocities in the case when the balls are moving in opposite directions.
Ans. $v = 1$, $v' = 8\frac{1}{2}$.

4. A mass of 50 lbs., moving at the rate of 10 feet per second, overtakes another mass of 25 lbs., moving at the rate of 6 feet per second; if both masses be perfectly elastic, find their velocities after the shock.

Ans. $v = 7\frac{1}{3}$; $v' = 11\frac{1}{3}$.

5. A sphere impinges directly on a sphere of the same mass. If they be both perfectly elastic, prove that they interchange velocities, after collision.

6. A mass drops from a height of 25 feet above a fixed horizontal plane, and rebounds to a height of 9 feet; find the coefficient of restitution. $e = \frac{3}{5}$.

7. A mass of 10 lbs., moving with a velocity of 10 feet per second, impinges directly on another of 5 lbs., supposed at rest. If the coefficient of restitution be $\frac{4}{5}$, and the duration of collision be $\frac{1}{100}$th part of a second, determine the *mean value* of the mutual pressure between the balls during the collision.

Here, we easily find $v' = 12$. Hence, by Art. 41, we find that the pressure which, acting for $\frac{1}{100}$th of a second, would generate a velocity of 12 feet per second, in a body of 5 lbs. mass, is $187\frac{1}{2}$ lbs.

8. An imperfectly elastic sphere falls from a given altitude above a horizontal plane, and rebounds continually: find the whole space described; and also the whole time before it is brought to rest; neglecting the time occupied by the series of impacts, and also the resistance of the air.

Let a denote the given altitude, and s the whole space described.

Then the height of the first rebound is ae^2; that of the second ae^4, &c.

Therefore $$s = a + 2ae^2 + 2ae^4 + \&c. = a\frac{1+e^2}{1-e^2}.$$

Again, if V, V_1, V_2, &c., V_n, be the velocities with which the sphere strikes the plane, at the different impacts, we have;

$$V_1 = eV, \quad V_2 = eV_1 = e^2V, \quad \&c., \; V_n = e^nV.$$

Also let t, t_1, t_2, &c., t_n, be the corresponding intervals of time; then

$$t = \frac{V}{g}, \quad t_1 = \frac{2V_1}{g} = \frac{2V}{g}e, \quad t_2 = \frac{2V}{g}e^2, \&c.$$

Hence, the entire time T is given by the equation

$$T = \frac{V}{g}(1 + 2e + 2e^2 + \&c.) = \frac{V}{g}\frac{1+e}{1-e}.$$

80. **Oblique Collision.**—We now proceed to the case of oblique collision, *i. e.* where the centres of the spheres are not moving in the same right line.

We shall suppose the spheres to be homogeneous, and perfectly smooth; so that their entire mutual action and reaction has place along the common normal at their point of contact, that is the line connecting the centres of the spheres. We also suppose that the lines along which the centres of the spheres are moving before collision lie in the same plane.

Let V, V' be the velocities at the instant of impact; and α, α' the angles which the line joining the centres, at the commencement of the collision, makes with the respective directions of motion.

Let v, v'; β, β' be the corresponding velocities and angles at the end of the collision.

Resolve V into its components, $V\cos\alpha$ and $V\sin\alpha$, respectively along and perpendicular to the right line joining the centres. Make a similar resolution of the velocities after collision. Then, since the forces brought into play during the collision act along the line joining the centres, the velocities perpendicular to that line are unaffected by the collision.

Hence we have

$$V\sin\alpha = v\sin\beta, \quad V'\sin\alpha' = v'\sin\beta'. \qquad (5)$$

Again, the component velocities $V\cos\alpha$, &c., along the line joining the centres of the spheres, will, by the Laws of Motion, be subject to the same relations (2), (3), as those already established for direct collision—hence, we obtain the two additional equations

$$v'\cos\beta' - v\cos\beta = e\,(V\cos\alpha - V'\cos\alpha'). \qquad (6)$$

$$MV\cos\alpha + M'V'\cos\alpha' = Mv\cos\beta + M'v'\cos\beta'. \qquad (7)$$

These, along with the two preceding equations (5), are sufficient for the determination of the velocities and the directions of motion after impact, when the corresponding velocities and directions before impact are known, as also the masses and the coefficient of restitution.

In the case of oblique collision of a sphere against a smooth fixed plane, let V be the velocity of the sphere before collision, and α the angle its direction of motion makes with the perpendicular to the smooth plane; then $V\cos\alpha$ represents the velocity perpendicular, and $V\sin\alpha$ that parallel, to the fixed plane.

If v and β represent the corresponding velocity and angle after collision, since the velocity parallel to the smooth plane is unaltered by collision, we have

$$v\sin\beta = V\sin\alpha. \qquad (8)$$

Again, the velocity perpendicular to the plane will be affected in the same manner as in direct collision, and we accordingly have

$$v \cos \beta = eV \cos \alpha, \tag{9}$$

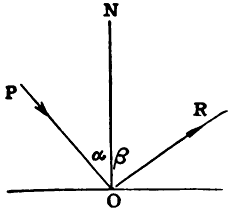
Hence, by division, we get

$$\tan \alpha = e \tan \beta, \tag{10}$$

which gives the direction of motion after impact.

The angles α, β are sometimes called the angles of incidence and reflexion; and the preceding result shows that the tangents of these angles are to each other in the constant ratio of the coefficient of restitution to unity. These angles are equal in the case of perfectly elastic bodies.

The subsequent motion of the body depends on the continuous forces which act on it. When gravity is the only acting force the path is a parabola, as in Art. 48; and the parabolic path is determined from the initial velocity v, and the initial direction of motion, β.



Examples.

1. If the mass M' be at rest before collision, find the directions of motion after collision.

$$Ans.\ \beta' = 0;\ \tan \beta = \frac{M + M'}{M - eM'} \tan \alpha.$$

2. A perfectly elastic ball impinges obliquely on another of equal mass at rest, prove that the directions of their motions after impact are at right angles to one another.

3. A ball impinges on another at rest; prove that if the coefficient of restitution be equal to the ratio of their masses the balls will move in directions at right angles to each other, whatever be the direction of the impact.

4. How is this statement to be modified in the case of direct collision?

The impinging ball is brought to rest by the collision.

5. A ball is reflected in succession by two fixed smooth planes of the same substance, which are at right angles to one another; the ball moves in a plane at right angles to the intersection of the fixed planes. Prove that the direction of motion before the first and after the second reflection are parallel.

6. A projectile strikes a perfectly elastic wall, which is perpendicular both to the horizon and to the plane of the projectile's flight: find the horizontal range of the reflected projectile.

Since the angles of incidence and reflection are equal in this case, as also the velocities before and after impact, it is evident that the parabolic path of the

projectile after striking the wall is equal in every respect to that which it would have continued to describe if there had been no wall interposed. Accordingly the problem is solved by aid of Art. 50.

7. An imperfectly elastic particle is projected from a point in a smooth horizontal plane, with a given velocity V, and in a given direction α, and proceeds to describe a series of parabolic paths by a number of rebounds from the plane: find the whole time elapsed before it ceases to rebound; and also its subsequent motion.

Resolve the velocity of projection into vertical and horizontal components, $V \sin \alpha$ and $V \cos \alpha$.

The horizontal component $V \cos \alpha$ will be unaltered by the successive impacts, and accordingly remains constant throughout the motion: the vertical component $V \sin \alpha$ will be altered at each impact in the same manner as in direct collision; accordingly it may be treated as in Ex. 8, p. 70.

Hence, if T be the entire time before the vertical velocity $V \sin \alpha$ is destroyed, we easily get, as in the example referred to,

$$T = \frac{2V \sin \alpha}{g} \frac{1}{1-e}.$$

Again, since the horizontal velocity is constant, and equal to $V \cos \alpha$, the whole range before the vertical velocity ceases is

$$\frac{V^2 \sin 2\alpha}{g(1-e)}.$$

The body would subsequently move along the plane with the constant velocity $V \cos \alpha$.

It should be observed that the particle, in this problem, describes a series of parabolic curves, one for each rebound.

81. **Vis Viva of a System.**—If each point of a mass m be moving with the same velocity, and if v denote, at any instant, the velocity common to all its points, then the quantity represented by mv^2 is called the *vis viva* of the mass at the instant. In general, in the motion of any system of masses, if each element of mass be multiplied by the square of its velocity at any instant, and the sum of these products taken for the entire system, this sum is called the *vis viva of the system* at that instant. It is represented by the expression

$$\Sigma (mv^2).$$

It is easily seen that, *in perfectly elastic Spheres, the Vis Viva is unaltered by Collision.*

For, since $e = 1$ in this case, equation (2) becomes $V + v = V' + v'$; also, we have

$$M(V - v) = M'(v' - V').$$

Multiplying, we get

$$M(V^2 - v^2) = M'(v'^2 - V'^2),$$

or
$$MV^2 + M'V'^2 = Mv^2 + M'v'^2. \qquad (11)$$

Hence, in direct collision between elastic spheres the *vis viva* is the same after collision as before.

It can be easily seen that the same principle holds in the oblique collision of perfectly elastic spheres. For the preceding demonstration holds for the components of velocity estimated along the line joining the centres of the spheres at the instant of collision: moreover, the tangential components of velocity are unaffected by collision; consequently (since $V^2 = V^2 \sin^2 a + V^2 \cos^2 a$, &c.) it follows, in the case of perfect elasticity, that the *vis viva* is unaltered by collision.

82. **Momentum of any System.**—Let (x, y, z), (x', y', z'), (x'', y'', z''), &c., at any instant, denote the coordinates of a number of moving particles, m, m', m'', &c., referred to a fixed system of rectangular axes; then, by Art. 12, the component velocities of m, at the instant, parallel to the axes of x, y, z, are $\dot{x}$, $\dot{y}$, $\dot{z}$, respectively.

Again, resolving the quantity of motion of m, in the same directions, we get for its components the expressions

$$m\dot{x},\ m\dot{y},\ m\dot{z}, \text{ or } m\frac{dx}{dt},\ m\frac{dy}{dt},\ m\frac{dz}{dt}.$$

If this be done for the other masses m', m'', &c., the whole quantity of motion or *momentum of the system, at the instant, estimated parallel to the axis of x*, is represented by $\Sigma m\,\dot{x}$.

In like manner, the whole momentum parallel to the axes of y and z are $\Sigma m\dot{y}$ and $\Sigma m\dot{z}$, respectively.

Again, let $\bar{x}$, $\bar{y}$, $\bar{z}$ represent the coordinates of the common centre of inertia of the system m, m', &c., at the same instant, we have, denoting the sum of the masses by M,

$$M\bar{x} = \Sigma mx, \quad M\bar{y} = \Sigma my, \quad M\bar{z} = \Sigma mz.$$

Moreover, since these equations are true throughout the

motion, we may differentiate them, with respect to the time t, and thus we obtain the equations

$$\left.\begin{aligned} M\frac{d\bar{x}}{dt} &= \Sigma m\frac{dx}{dt} = \Sigma m\dot{x} \\ M\frac{d\bar{y}}{dt} &= \Sigma m\frac{dy}{dt} = \Sigma m\dot{y} \\ M\frac{d\bar{z}}{dt} &= \Sigma m\frac{dz}{dt} = \Sigma m\dot{z} \end{aligned}\right\}. \tag{12}$$

Hence the resolved part of the momentum of a system in any direction is equal to the whole mass of the system multiplied into the component of the velocity of the centre of gravity, in the same direction.

83. **Conservation of Momentum.**—It is easily seen that the momentum in any direction of any system of bodies is unaltered by their mutual collision. For, under all circumstances of collision, the actions and reactions are equal and opposite; and as these forces are measured by the quantities of motion which they generate or destroy, it follows, whenever two bodies of the system come into collision, that whatever momentum, in any direction, is generated by the impact in one of the impinging bodies, an equal momentum in the same direction must be destroyed in the other. So that the entire momentum, in that direction, is unaltered by the collision. The same holds whatever number of collisions be supposed to take place between the members of the system.

Hence, we infer that the entire momentum of the system, resolved in any direction, is unaffected by impacts among the parts of the system.

It can be seen, without difficulty, that the same mode of reasoning applies to any case of internal mutual action between the several parts of the system, whether arising from attractions, molecular forces, or otherwise: since, in all cases, to every action corresponds an equal and opposite reaction.

We accordingly infer that, if a *system be subjected only to the internal mutual forces between the bodies which constitute it,*

the total resolved momentum in any direction is constant; i.e. $\Sigma m\dot{x}$, $\Sigma m\dot{y}$, $\Sigma m\dot{z}$, have constant values during the motion.

84. **Conservation* of Motion of Centre of Inertia.** —It follows from (12) that $\frac{d\bar{x}}{dt}, \frac{d\bar{y}}{dt}, \frac{d\bar{z}}{dt}$, i.e. the component velocities of the centre of inertia of a system, will be constant throughout the motion whenever the quantities of motion $\Sigma m\dot{x}$, $\Sigma m\dot{y}$, $\Sigma m\dot{z}$, are constant.

Hence, from the preceding Article, it follows that the velocity, and also the direction of motion of the centre of inertia of any system, are constant, whenever the system is subject only to the mutual actions and reactions of the bodies which constitute it.

This is a generalization of the principle of inertia contained in the First Law of Motion, and may be otherwise stated thus:—*A system of bodies cannot by their mutual actions and reactions alter the motion of their common centre of inertia.*

Hence, in such a system, when not acted on by any external forces, the common centre of inertia must either remain at rest or move uniformly in a right line.

85. The general principle, that the entire quantity of motion of two or more bodies is unaltered by their mutual actions and reactions, furnishes us with a ready method of solving some elementary problems.

For example, suppose two masses, m and m', to be connected by a string, and laid on a perfectly smooth horizontal table, at a distance from each other less than the length of the string. Now, let a given impulse be applied to m along the line which joins it to m', the motion which ensues after the string becomes stretched can be easily found as follows:

Let mV be the quantity of motion communicated to m by the impulse, then, after the string becomes tight, the bodies must move with a common velocity. Let v_1 denote this common velocity; then, since the whole quantity of motion of the two bodies remains the same, we have

$$(m + m')\,v_1 = mV.$$

* This proof corresponds in the main with that given by Newton. See Leges Motus, Cor. iv.

Consequently they move along the line with a common velocity

$$\frac{mV}{m + m'}.$$

In this problem we have supposed the motion one of pure sliding; and we neglect the mass of the string in it as also in the next problem.

86. *A mass M, after falling through a height h, from the edge of a smooth table, commences to draw by an inextensible string another mass M', which rests on the table; to find the velocity communicated to M' at the instant that the string becomes tightened, and also the impulse of the tension of the string.*

The velocity acquired by M, in consequence of its fall, is represented by $\sqrt{2gh}$; and since at the end of the impulsive strain the bodies are moving with equal velocities, and also the quantity of motion is unaltered by the impulsive action, we must have

$$(M + M')v_1 = MV = M\sqrt{2gh},$$

or
$$v_1 = \frac{M}{M + M'}\sqrt{2gh}, \tag{13}$$

where v_1 denotes the common velocity at the instant in question.

Again, the impulse of the tension of the string is measured by the quantity of motion communicated to M'; and accordingly is represented by

$$\frac{MM'}{M + M'}\sqrt{2gh}.$$

If the table be rough, since the friction of the table is proportional to the weight of M', it may be neglected in comparison with the impulsive force, and we obtain the same value for v_1 as in the case of a smooth table (see Art. 67).

Examples.

1. A sphere, of 30 lbs., moving with a velocity of 45 feet a second, overtakes another, of 27 lbs., moving 32 feet a second; if the relative coefficient of restitution be $\frac{2}{3}$, find their velocities after collision. *Ans.* $34\frac{14}{19}$, $43\frac{23}{57}$.

2. Two spheres meet directly with equal velocities; find the ratio of their masses that one of them, M, should be reduced to rest by the collision—(1) when perfectly elastic; (2) for coefficient of restitution e.

Ans. (1) $M = 3M'$, (2) $M = M'(1 + 2e)$.

3. If two equal and perfectly elastic spheres be dropped at the same instant from different heights, h and h', above a horizontal plane; determine whether their common centre of inertia will ever rise to its original height.

Ans. No, unless $\sqrt{\frac{h}{h'}}$ is a commensurable number.

4. A 10 lb. shot is fired from a gun of 12 cwt., that is quite free to move. The velocity with which the shot leaves the mouth of the gun is 1600 feet per second; find the velocity of the gun's recoil. *Ans.* 11·9 feet per second.

5. Three homogeneous spherical bodies, m, m', m'', are placed with their centres in a row. If m be projected with a given velocity V towards m'; to find the magnitude of m' in order that the velocity communicated to m'' by its intervention shall be the greatest possible.

Let e, e' denote the relative coefficients of restitution between m, m', and between m', m'', respectively. Then, if v' be the velocity of m' after the first collision, we get from Art. 78,

$$v' = \frac{m(1+e)V}{m+m'}.$$

In like manner, if v'' be the velocity of m'' after the second collision, we have

$$v'' = \frac{m'(1+e')v'}{m'+m''} = \frac{mm'(1+e)(1+e')V}{(m+m')(m'+m'')}.$$

Accordingly, $\dfrac{m'}{(m+m')(m'+m'')}$ must be a maximum; or $\dfrac{(m+m')(m'+m'')}{m'}$ is a minimum: i.e. $m + m' + m'' + \dfrac{mm''}{m'}$ is a minimum, or $m' + \dfrac{mm''}{m'}$ is a minimum: hence, $m' = \sqrt{mm''}$, by elementary algebra; consequently the masses must be in geometrical progression.

This reasoning is readily extended to the case of any number of spheres placed in a row; and, when the first and last are given, the masses must be in geometrical progression, in order that the velocity communicated to the last should be the greatest possible.

6. Two particles are connected by a string, and laid on a uniformly rough horizontal table, at a distance from each other less than the length of the string. One of the particles receives a given impulse along the line joining them: determine the motion which ensues after the tightening of the string.

7. Find an expression for the *vis viva* lost in the direct collision of two imperfectly elastic spheres.

From equations (2), (3), Art. 78, we have

$$(mV + m'V')^2 = (mv + m'v')^2,$$

and

$$mm'e^2(V - V')^2 = mm'(v - v')^2.$$

This latter may be written

$$mm'(V - V')^2 = mm'(v - v')^2 + (1 - e^2)mm'(V - V')^2.$$

Hence, by addition,

$$(m + m')(mV^2 + m'V'^2) = (m + m')(mv^2 + m'v'^2) + (1 - e^2)mm'(V - V')^2.$$

Therefore $$mV^2 + m'V'^2 = mv^2 + m'v'^2 + (1 - e^2)\frac{mm'}{m + m'}(V - V')^2.$$

Accordingly, the *vis viva* lost by the collision is represented by

$$(1 - e^2)\frac{mm'}{m + m'}(V - V')^2.$$

8. Find the loss of *vis viva* caused by the direct impact of two balls, one weighing 10 lbs. and falling from a height of 20 feet, the other at rest and weighing 30 lb.; assuming the coefficient of restitution $= \frac{1}{2}$.

Ans. $\frac{9}{16}$th of the original *vis viva*.

9. A body, after sliding down a smooth inclined plane of given height, rebounds from a hard horizontal plane; find the range on the latter plane.

10. A mass M, after falling freely through h feet, begins to pull up a heavier mass M_1 by means of a string passing over a pulley, as in Atwood's machine; find the height through which it will lift it.

Let v_1 be the velocity communicated to M_1 by the impulsive action; then by Art. 86 we have $v_1 = \frac{M}{M + M_1}\sqrt{2gh}$.

During the subsequent motion M_1 is subject to a uniform retardation $\frac{M_1 - M}{M_1 + M}g$, as in Art. 73; accordingly, if H denote the height to which M_1 ascends before it is brought to rest, we have

$$H = \frac{v_1^2}{2f} = \frac{M^2}{M_1^2 - M^2}h.$$

11. An inelastic particle falls from rest to a fixed inclined plane, and slides down the plane to a fixed point in it; show that the locus of the starting point is a straight line when the time to the fixed point is constant. (*Camb. Trip.*, 1871).

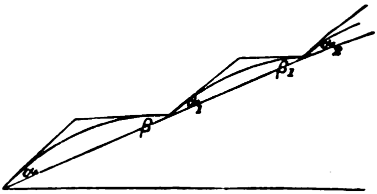
12. Two equal balls of radius a are in contact and are struck simultaneously by a ball of radius c moving in the direction of their common tangent; if all the balls be of the same material, the coefficient of elasticity being e, find the velocities of the balls after impact, and prove that the impinging ball will be reduced to rest if

$$2e = \frac{c^2(a+c)^2}{a^3(2a+c)}. \qquad (\textit{Camb. Trip.}, 1871.)$$

13. Show how to determine the motion of two elastic spheres after direct impact, and prove that the relative velocity of each of them with regard to the centre of mass of the two is, after the impact, reversed in direction and reduced in the ratio $e : 1$, e being the coefficient of restitution.

A series of n elastic spheres whose masses are 1, e, e^2, &c., are at rest, separated by intervals, with their centres on a straight line. The first is made to impinge directly on the second with velocity u. Prove that the final *vis viva* of the system is $(1 - e + e^n)u^2$. (*Ibid.*, 1875.)

14. An elastic ball makes a series of rebounds from a perfectly smooth inclined plane: to investigate its motion.



Let i be the inclination of the plane to the horizon, and suppose the ball projected from the point O in the plane, in a direction which makes the angle α with the plane. Let β, β_1, &c., β_n be the angles at which the ball strikes the plane at the first, second, . . . n^{th} impacts; and α_1, α_2, . . . α_n, the angles it makes after rebounding.

Then, by equation 9, Art. 56, we have

$$\cot\beta = \cot\alpha - 2\tan i;$$

but, by (10) Art. 80,

$$\cot\beta = e\cot\alpha_1,$$

$$\therefore\quad e\cot\alpha_1 = \cot\alpha - 2\tan i.$$

Similarly

$$e\cot\alpha_2 = \cot\alpha_1 - 2\tan i;$$

$$\therefore\quad e^2\cot\alpha_2 = \cot\alpha - 2(1+e)\tan i;$$

and it is easily seen that we have, in general,

$$e^n\cot\alpha_n = \cot\alpha - 2(1 + e + \ldots + e^{n-1})\tan i$$

$$= \cot\alpha - \frac{2(1-e^n)}{1-e}\tan i,$$

from which the angle after the n^{th} rebound can be found.

Again, the ball will proceed to bound up the plane so long as the angles α_1, α_2, . . . are each less than $90° - i$.

If a_n be the first of a series of angles which exceeds $90° - i$, we will have $\cot a_n < \tan i$.

If $\cot a$ is greater than $\frac{2 \tan i}{1-e}$ it can be readily shown that for all values of n a_n is less than $90° - i$; and in this case accordingly the ball would proceed to ascend the plane by an indefinite series of parabolic paths.

But if $\cot a$ be less than $\frac{2 \tan i}{1-e}$, after a certain number of impacts, the body would proceed to rebound *down* the inclined plane.

In the particular case where $\cot a = \frac{2 \tan i}{1-e}$, or $2 \tan i = \cot a (1-e)$, we have

$$e \cot a_1 = e \cot a ; \quad \therefore a_1 = a ;$$

hence

$$a = a_1 = a_2 = \ldots = a_n ;$$

or, all the angles of rebound are equal to one another; consequently the series of parabolic paths in this case are similar, and the particle would proceed up the plane with an indefinite number of rebounds.

In general, let $t_1, t_2, \ldots t_n$ be the times of flight for the series of parabolic paths, and $v_1, v_2, \ldots v_n$, the velocities of the successive rebounds; then by equation (5), Art. 50, we have

$$t_1 = \frac{2v \sin a}{g \cos i}, \quad t_2 = \frac{2v_1 \sin a_1}{g \cos i}, \text{ \&c.}$$

But if v' be the velocity with which the ball strikes the plane at the first impact, we have

$$v' \sin \beta = v \sin a ;$$

but by Art. 80,

$$v_1 \sin a_1 = ev' \sin \beta = ev \sin a ;$$

consequently

$$t_2 = et_1 ; \text{ also } t_3 = et_2 = e^2 t_1, \text{ \&c.}$$

Hence the times of flight are in geometrical progression, having e for their common ratio.

If the intervals of time occupied by the successive impacts be neglected, we get for the time T of describing the first n parabolas,

$$T = \frac{2v \sin a}{g \cos i} \, \frac{1-e^n}{1-e}.$$

Again, let $R_1, R_2, \ldots R_n$ denote the consecutive ranges on the inclined plane; then, by Art. 50, we have

$$R_1 = \tfrac{1}{2} g t_1^2 \frac{\cos (a+i)}{\sin a} = \tfrac{1}{2} g t_1^2 \cos i (\cot a - \tan i).$$

Similarly,

$$R_2 = \tfrac{1}{2} g t_2^2 \cos i \,(\cot a_1 - \tan i) = \tfrac{1}{2} g t_1^2 \cos i \,(e^2 \cot a_1 - e^2 \tan i)$$

$$= \tfrac{1}{2} g e t_1^2 \cos i \,\{\cot a - (2+e) \tan i\}.$$

And, in general,

$$R_n = \tfrac{1}{2}gt_n^2 \cos i\,(\cot \alpha_{n-1} - \tan i)$$

$$= \tfrac{1}{2}ge^{n-1}\,t_1^2 \cos i\,\{\cot \alpha - (2 + 2e + \ldots + 2e^{n-2} + e^{n-1}) \tan i\}$$

$$= \tfrac{1}{2}ge^{n-1}\,t_1^2 \cos i \cot \alpha - \tfrac{1}{2}ge^{n-1}\,t_1^2 \sin i \left(\frac{2 - e^{n-1} - e^n}{1 - e}\right)$$

$$= \frac{2v^2 \sin^2\alpha}{g \cos i}\left\{e^{n-1} \cot \alpha - \frac{2e^{n-1}}{1 - e} \tan i + \frac{1 + e}{1 - e} e^{2n-2} \tan i\right\}.$$

Hence the sum of n ranges is found to be

$$= 2\frac{v^2}{g}\frac{\sin^2\alpha}{\cos i}\frac{1 - e^n}{1 - e}\left\{\cot \alpha - \frac{1 - e^n}{1 - e} \tan i\right\}$$

$$= vT \sin \alpha \left(\cot \alpha - \frac{1 - e^n}{1 - e} \tan i\right).$$

If $\cot \alpha$ be greater than $\dfrac{2 \tan i}{1 - e}$ we get the entire range on the inclined plane by making $n = \infty$; hence the entire range is, in this case,

$$vT \sin \alpha \left\{\cot \alpha - \frac{\tan i}{1 - e}\right\}.$$

The preceding question was discussed at great length by Bordoni, *Mem. della Societa Ital.*, 1816. See also Walton's *Problems on Theoretical Mechanics*, pp. 262, 263, 3rd edition.

15. In the preceding example show that the greatest distances of the body from the inclined plane in the successive parabolic paths are in geometrical progression, having e^2 as their common ratio.

16. If two bodies, of the same elasticity be projected with the same velocity from a point on an inclined plane, and if the directions of projection make equal angles at opposite sides of the perpendicular to the plane, prove that the series of parabolic paths described, one up, the other down the plane, will be described in times which are respectively equal in pairs.

17. An imperfectly elastic ball falls from a height h upon an inclined plane; find the range between the first and second rebounds. *Ans.* $4eh \sin i\,(1 + e)$.

18. Prove that, in order to produce the greatest deviation in the direction of a smooth billiard ball of diameter a, by impact on another equal ball at rest, the former must be projected in a direction making an angle $\sin^{-1} \dfrac{a}{c}\sqrt{\dfrac{1 - e}{3 - e}}$ with the line (of length c) joining the two centres; e being the coefficient of restitution.
Camb. Trip., 1873.

19. A bucket and a counterpoise, connected by a string passing over a pulley, just balance one another, and an elastic ball is dropped into the centre of the bucket from a distance h above it; find the time that elapses before the ball

ceases to rebound; and prove that the whole descent of the bucket during this interval is $\frac{4mh}{2M+m}\,\frac{e}{(1-e)^2}$ where m, M are the masses of the ball and the bucket, and e is the coefficient of restitution. *Camb. Trip.*, 1875.

Let v be the velocity of the ball just before the first impact. The relative velocity after the first impact is ev, and the relative acceleration is g, since the acceleration of the bucket is zero.

Therefore the time during which the ball rebounds is

$$\frac{2v}{g}(e+e^2+e^3+\dots)=\frac{2v}{g}\,\frac{e}{1-e}=2\,\frac{e}{1-e}\sqrt{\frac{2h}{g}}.$$

Let V_1, V_2, V_3, ... be the velocities of the bucket between the first, second, third, ... impacts.

Then $$V_1=\frac{m(1+e)}{2M+m}v,\quad V_2=V_1+\frac{m(1+e)}{2M+m}ev,\ \&c.,$$

and the space described by the bucket is

$$\frac{2v}{g}(eV_1+e^2V_2+e^3V_3+\dots)=\frac{2mev^2}{g(2M+m)(1-e)^2}=\frac{4mh}{2M+m}\,\frac{e}{(1-e)^2}.$$

(This proof is taken from Greenhill's solutions of Cambridge Problems and Riders for 1875.)

20. A particle is projected with a velocity V, in a direction making an angle α with the horizon, and strikes a vertical wall, at a distance a from the point of starting. Find when and where it will strike the horizontal plane drawn through its initial position.

Ans. $T=\frac{2V\sin\alpha}{g}$. The distance from the wall at which it will strike the ground $=e\left(\frac{V^2\sin 2\alpha}{g}-a\right)$, where e is the coefficient of restitution for the particle and the wall.

21. A large number of equal particles are fastened at unequal intervals to a fine string, and then collected into a heap at the edge of a smooth horizontal table, with the extreme one just hanging over the edge. The intervals are such that the times between successive particles being carried over the edge are equal: prove that if c_n be the interval between the n^{th} and the $(n+1)^{th}$ particle, and v_n the velocity just after the $(n+1)^{th}$ particle is carried over, then $\frac{c_n}{c_1}=\frac{v_n}{v_1}=n$.

Professor Wolstenholme, *Educ. Times.*

If v be the velocity acquired by the first particle during its fall through the interval c_1, we get immediately, from the conditions of the problem, the two series of relations

$$v_1=\tfrac{1}{2}v,\quad v_2=\tfrac{2}{3}(v_1+v)=\tfrac{2}{2}v,\quad v_3=\tfrac{3}{4}(v_2+v)=\tfrac{3}{2}v,\ \&c.$$

$$2gc_1=v^2,\quad 2gc_2=(v_1+v)^2-v_1^2=2v^2,\quad 2gc_3=(v_2+v)^2-v_2^2=3v^2,\ \&c.$$

Hence

$$v_1:v_2:v_3:\&c.:v_n=c_1:c_2:c_3:\&c.:c_n=1:2:3:\&c.:n.$$

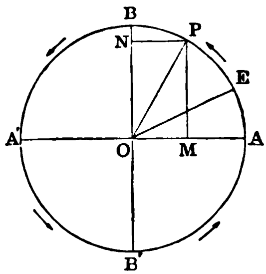
CHAPTER V.

CIRCULAR MOTION.

SECTION I.—*Harmonic Motion.*

87. **Uniform Circular Motion.**—If a point P describe a circle with a uniform motion, the radius of the circle is called the *amplitude* of the motion, and the time of making one revolution is called its *period*. If the arcs are measured from a fixed point A, and the time counted from the instant the moving point passed through a fixed point E, then the angle AOE is called the angle of epoch, or briefly, *the epoch*. Also the ratio which the arc PE, at any instant, bears to the circumference of the circle is called the *phase* of the moving point at that instant.

The arrowheads on the figure denote the direction in which the motion is supposed to take place, and such a rotation as there represented, *i.e.* in the opposite direction to that of the hands of a clock, is considered a *positive* rotation: that in the opposite direction, or *clockwise*, being considered negative.



Let ω be the angular velocity of P, or the circular measure of the arc described in one second, ϵ the circular measure of the epoch AOE, and θ that of AOP, we have

$$\theta = \omega t + \epsilon. \qquad (1)$$

Again, if T denote the period, we get $\omega = \frac{2\pi}{T}$, and hence, if desirable, we should write

$$\theta = \frac{2\pi}{T} t + \epsilon;$$

but we shall generally employ the form $\theta = \omega t + \epsilon$, being more compendious.

88. **Harmonic Motion.**—If PM be drawn perpendicular to the diameter AA', then as P moves uniformly round the circle, the point M moves backwards and forwards along the line AA', and is said to have a *simple harmonic motion.* The amplitude, period, epoch, and phase of the harmonic motion are the same as those of the corresponding circular motion.

If $OM = x$, then the position of M at any instant is given by the equation

$$x = a \cos (\omega t + \epsilon), \tag{2}$$

where a represents the amplitude, and ϵ the epoch of the motion. The angle $\omega t + \epsilon$ is called the *argument* of the motion, and the distance x is said to be a *simple harmonic function of the time.*

Again, if PN be perpendicular to OB, and $y = ON$, we have

$$y = a \sin (\omega t + \epsilon) = a \cos (\omega t + \epsilon - \tfrac{1}{2}\pi).$$

Hence the point N has also a harmonic motion, and we infer that a uniform circular motion is equivalent to two simultaneous rectangular harmonic motions, of the same amplitude and period, but differing one-fourth in phase: and *conversely.*

Again, if the point M be projected on any line, the projected point plainly has a harmonic motion of the same period and phase, but having for amplitude the projection of the amplitude of M.

If we differentiate equation (2) we get

$$v = \frac{dx}{dt} = - a\omega \sin (\omega t + \epsilon).$$

Consequently the velocity of a point which has a simple harmonic motion is a simple harmonic function of the time; and its maximum value is equal to the velocity in the circle.

Again, the acceleration f is given by the equation

$$f = \frac{dv}{dt} = - \omega^2 a \cos (\omega t + \epsilon) = - \omega^2 x.$$

Consequently the acceleration at any instant is proportional to the distance from the middle point of the motion, and is always directed towards that point. The acceleration at either extremity of the motion is $- \omega^2 a$.

Any number of harmonic motions of equal periods in the same line are equivalent to a single harmonic motion.

For let $x = a\cos(\omega t + \epsilon) + a'\cos(\omega t + \epsilon') + \&c.$

Then $$x = A\cos\omega t - B\sin\omega t,$$

where $$A = \Sigma a\cos\epsilon, \quad \text{and} \quad B = \Sigma a\sin\epsilon.$$

Hence $x = C\cos(\omega t + \gamma)$, where

$$C = \sqrt{A^2 + B^2}, \text{ and } \tan\gamma = \frac{B}{A}.$$

This result admits also of a simple geometrical demonstration.

89. **Elliptic Harmonic Motion.**—If a circle be projected orthogonally on any plane its projection is an ellipse, and the projection of any point which moves uniformly on the circle is said to have an *elliptic harmonic motion.*

An elliptic harmonic motion may be resolved into two simple harmonic motions, of the same period but differing in amplitude, along any two conjugate diameters of the ellipse, these motions differing one-fourth in phase. This follows immediately from the property that rectangular diameters in the circle are projected into conjugate diameters in the ellipse. Conversely, any two simple harmonic motions, in different lines, of the same period and differing one-fourth in phase, compound an elliptic harmonic motion, having the lines for conjugate diameters.

Examples.

1. A point P describes a circle with uniform velocity. If M be its projecture on any fixed diameter, prove that the velocity of M varies as PM, and that its acceleration varies as OM; O being the centre of the circle.

2. If two harmonic motions in the same line have equal amplitude (a) and equal periods, but different epochs, ϵ, ϵ', find the amplitude of their resultant motion. *Ans.* $2a\cos\frac{1}{2}(\epsilon - \epsilon')$.

3. If the difference of phase in the last passes continuously from 0 to 2π, find the mean value of the square of the amplitude of the resulting vibration. *Ans.* $2a^2$.

The mean value is represented by the definite integral (*Int. Calc.*, Art. 238),

$$\frac{8a^2}{\pi}\int_0^{\frac{\pi}{2}} \cos^2\phi \, d\phi.$$

This result is of importance in the Wave Theory of Light, as it shows that the intensity of light is proportional to the square of the amplitude of the vibration which constitutes the light.

4. If two or more harmonic motions in different directions have the same periods and phases, show that their resultant is also a simple harmonic motion of the same phase.

5. Prove that the resultant of any number of simple harmonic motions, differing in directions and phases, but having the same period, is an elliptic harmonic motion.

6. In elliptic harmonic motion prove that the areal velocity of the moving point, round the centre, is constant.

7. Prove that any simple harmonic motion is equivalent to two circular vibrations, in opposite directions.

8. A horizontal shelf moves vertically with simple harmonic motion, the complete period being one second. Find the greatest amplitude in centimetres that objects resting on the shelf may remain in contact with it when at its highest point: assuming $g = 981$. *Ans.* 24·85.

9. In elliptic harmonic motion, being given the difference of phase, and the ratio of the amplitudes of the components along two given right lines, perpendicular to each other, determine the position and the ratio of the axes of the ellipse.

If k be the ratio of amplitudes, $\frac{\epsilon}{2\pi}$ the difference of phase, μ the ratio of axes, and α the angle made by the axis major with the direction of greater amplitude, then

$$\mu^2 = \frac{1 + k^2 - \sqrt{1 + 2k^2\cos 2\epsilon + k^4}}{1 + k^2 + \sqrt{1 + 2k^2\cos 2\epsilon + k^4}}, \quad \tan 2\alpha = \frac{2k\cos\epsilon}{1 - k^2}.$$

10. Show that two simple harmonic motions, in rectangular directions, of the same epoch, and whose periods are as 1 : 2, compound a parabolic vibration.

In this case the motion may be represented by the equations

$$x = a\cos 2\omega t, \quad y = b\cos\omega t.$$

Hence, eliminating t, we get $\quad \dfrac{x}{a} = \dfrac{2y^2}{b^2} - 1.$

11. In the same case, if the vibrations differ in epoch, show that the harmonic motions compound a curve of the fourth degree.

The motion is represented by the equations

$$x = a\cos(2\omega t - \epsilon), \quad y = b\cos\omega t.$$

Hence, eliminating t, we get

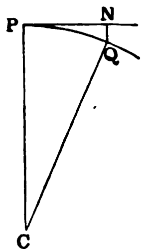
$$\frac{x^2}{a^2} + \frac{4y^2}{b^2}\left(\frac{y^2}{b^2} - 1\right) + \frac{2x}{a}\left(1 - \frac{2y^2}{b^2}\right)\cos\epsilon + \cos^2\epsilon = 0.$$

Section II.—*Centrifugal Force.*

90. **Centrifugal and Centripetal Force.**—A heavy particle may be made to move in a circle, either by having it attached to a fixed point by an inextensible string, and made to move on a plane passing through the fixed point, or by its being constrained to move in a fixed circular groove. During the motion in the former case the string sustains a strain or tension: in the latter, the moving particle presses outwardly against the groove. This tension, or pressure, is called the centrifugal force of the particle, and is always directed outwards from the centre of the circle described.

The groove, or string, exerts at the same instant an equal and opposite reaction, inwards on the particle. This latter is called the centripetal force which acts on the particle. If m be the mass of the particle, and V its velocity at any instant, then, by Art. 25, we infer that the centripetal force is represented by $m\dfrac{V^2}{r}$, where r is the radius of the circle.

This result can also be established otherwise in the following manner. Let P be the position of the particle at any instant; then if it were free, and acted on by no force, it would move along the tangent at P with the velocity V, which it has at the instant; and at the end of the time Δt would arrive at the point N, where $PN = V \times \Delta t$: accordingly QN denotes the space through which it has moved, in the time Δt, owing to the centripetal force. This force is directed towards the centre of the circle, and may be regarded as constant in magnitude and direction, during the indefinitely short time Δt.



Hence, if f denote its acceleration, we have, by Art. 36,

$$QN = \tfrac{1}{2} f (\Delta t)^2.$$

But, in the limit, $PN^2 = 2QN.PC$,

where C is the centre of the circle;

$$\therefore \quad V^2 (\Delta t)^2 = 2QN.PC.$$

Hence
$$f = \frac{V^2}{r}. \tag{1}$$

Or, *the centrifugal acceleration f is a third proportional to the radius of the circle and the velocity of the particle.*

The centrifugal force is accordingly represented by $\frac{mV^2}{r}$.

If it be required to calculate the pressure in pounds due to the centrifugal force, we substitute $\frac{W}{g}$ for m, and the preceding expression becomes $\frac{W}{g}\frac{V^2}{r}$.

Since the centripetal force is always directed to the centre of the circle, and is consequently at right angles to the direction of motion, it has no effect in altering the velocity of the moving particle. Hence, if no other force act on the particle, its velocity will be constant during the motion.

Conversely, if a particle m describe a circle of radius r, with a uniform velocity V, we infer that the resultant of all the forces which act on it passes through the centre of the circle, and is represented by $\frac{mV^2}{r}$.

Again, as the velocity in the circle is uniform, if T denote the number of seconds in which the circumference is described, we have $V = \frac{2\pi r}{T}$.

Hence, in this case, we have
$$f = 4\pi^2 \frac{r}{T^2}. \tag{2}$$

Consequently, in uniform circular motion, the centrifugal force varies directly as the radius of the circle, and inversely as the square of the time of revolution.

Again, if ω be the angular velocity of the radius CP, we have $\omega = \frac{2\pi}{T}$: accordingly, in terms of the angular velocity and the distance, we have
$$f = \omega^2 r. \tag{3}$$

EXAMPLES.

1. Calculate the centripetal acceleration of a particle which moves in a circle of 5 feet radius with a velocity of 10 feet per second. *Ans.* 20.

2. A particle performs 20 revolutions per minute in a circle of 1 foot circumference: find its centrifugal acceleration. *Ans.* ·6981.

3. A body of 1 lb. mass revolves, in a horizontal plane, at the extremity of a string 10 feet long, so as to make a complete revolution in 2 seconds; find the tension of the string in pounds. *Ans.* $\frac{10\pi^2}{g}$.

4. A railway carriage of 1 ton weight is moving at the rate of 60 miles an hour round a curve of 1210 feet radius: find the centrifugal pressure on the rails. *Ans.* 448 lbs.

5. In the last example, find how much the outer rail should be raised in order that the total pressure should be equally distributed between both the rails, the distance between the rails being 4 feet. *Ans.* $9\frac{1}{2}$ inches, approximately.

91. **Circular Orbits.**—In the case of uniform circular motion, since the centrifugal force acting on the particle at each instant is directed from the centre of the circle, we may suppose the particle to be kept in its circular orbit by the action of an attractive force always directed to the centre of the circle, and whose acceleration is $4\pi^2 \frac{r}{T^2}$. Hence, if the magnitude of the acceleration directed to a fixed centre of force be known, we can determine the conditions that a particle should describe a circle, having the fixed point as its centre. For, if f be the acceleration caused by the central force at the distance r of the particle, we have $f = \frac{v^2}{r}$, and therefore $v = \sqrt{fr}$. This determines the velocity at each point in the circle.

Conversely, if the particle be projected at the distance r from the centre of force, at right angles to the radius vector, with a velocity $v = \sqrt{fr}$, it will proceed to describe a circle freely round the centre of force.

Also, the time T of describing the circle will be $\frac{2\pi r}{v}$, or

$$T = 2\pi\sqrt{\frac{r}{f}}.$$

For example, if the attractive force be directly proportional to the distance, we have $f = \mu r$, where μ is some constant; and consequently, in this case,

$$T = \frac{2\pi}{\sqrt{\mu}}. \tag{4}$$

Hence, for this law of attraction the time of revolution in a circular orbit is independent of the distance; and we infer that the times of revolution for all circular orbits round the same centre of force are equal.

Again, let the attraction vary inversely as the square of the distance from the centre, that is to say let $f = \frac{\mu}{r^2}$.

In this case the velocity in the circular orbit is represented by $\sqrt{\frac{\mu}{r}}$, and the time of revolution by $2\pi\sqrt{\frac{r^3}{\mu}}$. Hence we see that in different circular orbits round the same centre of force (which varies as the inverse square of the distance), the *squares of the periodic times vary as the cubes of the distances from the centre of force.*

This establishes Kepler's Third Law for circular orbits.

The preceding are particular cases of general results connected with the problem of Central Forces, which will be treated of in a subsequent Chapter.

92. **Centrifugal Force at Earth's Equator.**—We now proceed to consider the centrifugal force arising from the rotation of the Earth on its axis.

Let r be the number of feet in the Earth's radius; T the number of seconds in the time of a complete rotation on its axis; f the acceleration due to centrifugal force at the Equator; then we have

$$f = 4\pi^2 \frac{r}{T^2}.$$

The most convenient method of determining f is by comparing its value with that of g at the Equator: thus

$$\frac{f}{g} = \frac{4\pi^2 r}{gT^2}. \tag{5}$$

Substituting their numerical values for π, r, g, and T, we find, to the nearest integer,

$$g = 288f.$$

Again, as the centrifugal force tends to diminish the action of gravity, it should be added to the observed value of g to obtain the true acceleration due to the Earth's attraction at the Equator.

Hence we have $\quad G = g + f = 289f,$

$$\therefore \quad f = \frac{G}{289}, \tag{6}$$

or, the centrifugal force at the Equator is the 289th part of the Earth's attraction at the same place, approximately.

It is easy from this result to determine what the time of the Earth's rotation should be in order that bodies should have no weight at the Equator.

For let T' be the time required, then the corresponding centrifugal acceleration would be $\frac{4\pi^2 r}{T'^2}$;

hence $$\frac{4\pi^2 r}{T'^2} = G = 289 \cdot \frac{4\pi^2 r}{T^2};$$

$$\therefore T' = \frac{T}{17}. \tag{7}$$

Accordingly, if the Earth were to rotate 17 times faster than it does bodies would lose all their weight at the Equator.

Hence also we infer that if a body revolve around the Earth in a circle, near its surface, and subject to its attraction solely, it should travel round the circumference of its circular orbit in the 17th part of a day.

93. **Verification of the Law of Attraction.**—The last result can be applied to verify, by a rough calculation, the fact that the Moon is retained in her orbit by the attractive force of the Earth; the law of force being the inverse square of the distance from the Earth's centre; and the Moon's path being assumed to be a circle having the centre of the Earth as its centre.

For let R denote the distance of the centre of the Moon from the Earth's centre; T her periodic time expressed in days; T' that of a body revolving round the Earth close to its surface.

Then, by Art. 91, we have

$$T^2 : T'^2 = R^3 : r^3. \tag{8}$$

Now, assuming the Moon's distance from the Earth's centre to be 60 times the Earth's radius, as found approximately by observation, we have $R = 60\,r$.

Hence $$T = T'\,60^{\frac{3}{2}} = 60T'\sqrt{60};$$

but, by the last Article, $T' = \dfrac{1}{17}$ (since the times T, T' are expressed in days).

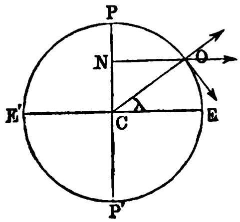
Hence $$T = \frac{60}{17}\sqrt{60}.$$

This, when calculated, gives approximately

$$T = 27{\cdot}3387 \text{ days}.$$

The near agreement (within 24 minutes) of this result with the mean value of T as obtained by observation, viz., 27·3216 days, affords a strong confirmation of the Law of Gravitation. When more accurate values are substituted, and all the circumstances of the problem taken into account, the calculated agrees completely with the observed result.

94. **Tangential and Normal Components.**—Let O represent any place, of latitude λ, on the Earth's surface, supposed spherical: ON the perpendicular drawn from O to the Earth's axis, PP'.



Then the centrifugal acceleration at O is in the direction NO produced; and its amount is represented by

$$\frac{4\pi^2 NO}{T^2}, \text{ or } \frac{4\pi^2 r\cos\lambda}{T^2} = f\cos\lambda,$$

where f denotes the centrifugal acceleration at the Equator.

The centrifugal force along NO can be resolved into two components; one along CO produced, the other in the tangential direction.

These are plainly represented by $f\cos^2\lambda$, and $f\cos\lambda\sin\lambda$, respectively; or by $\frac{G\cos^2\lambda}{289}$, and $\frac{G\cos\lambda\sin\lambda}{289}$. The effect of the former is to diminish the Earth's attraction as before.

Hence for the actual value of g at any latitude λ, assuming the Earth an exact sphere, we get

$$g = G - \frac{G\cos^2\lambda}{289} = G\left(1 - \frac{\cos^2\lambda}{289}\right). \qquad (9)$$

This result has to be modified when the spheroidal form of the Earth is taken into account.

The tangential component of the centrifugal force vanishes at the Equator and also at the poles. For intermediate places it varies as $\sin 2\lambda$, and has its greatest value at 45° latitude, where it is equal to half the centrifugal acceleration at the Equator.

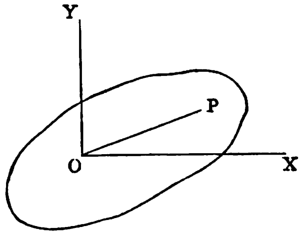
Examples.

1. Calculate the diminution of gravity due to centrifugal force at a latitude of 45°.
2. Calculate the tangential component for the same case.
3. Calculate the centrifugal acceleration at the equator of the planet Mercury, its radius being 1570 miles, and time of revolution $24^h\ 5^m$. *Ans.* ·0435.

95. **Rotation of a Rigid Body.**—If a rigid body be conceived to turn round a fixed axis, each of its points will describe a circle, having its centre on the axis of revolution. Also, since every line in the body that is perpendicular to the axis turns through the same angle, the angular velocity of each point of the rigid body will be the same at any instant. This instantaneous angular velocity is called the *angular velocity of the body*, and it is plainly the same as the velocity of any point in the body which is at the unit of distance from the fixed axis.

If the angular velocity at any instant be represented by ω, the velocity of a point whose distance from the axis is p is represented by $p\omega$.

96. *If a plane lamina rotate about a fixed axis at right angles to its plane, the centrifugal forces of the different elements of the lamina are equivalent to a single force, passing through its centre of mass, and which is the same as if the entire mass were concentrated at that point.*



Let the plane of the paper represent that of the lamina; and take the point O, in which the fixed axis meets the plane, as the origin of a pair of *fixed* rectangular axes OX and OY.

Suppose ω to be the angular velocity of the lamina at any instant; then, since each point in the lamina describes a circle round O, the centrifugal forces for all its elements pass through that point; these forces accordingly are equivalent to a single force. To find the value of this single resultant; let $OP = r$, and let dm denote the mass of an element at the point P; then the centrifugal force of the element is $\omega^2 r\,dm$, acting along the line OP produced. This force can be decomposed into two, $\omega^2 x\,dm$ and $\omega^2 y\,dm$, parallel to the axes of x and y respectively.

Suppose the centrifugal forces of the other elements resolved in like manner, then the entire system is equivalent to the forces $\omega^2 \Sigma x\,dm$ and $\omega^2 \Sigma y\,dm$, parallel to OX and OY. But

$$\Sigma x\,dm = M\bar{x}, \quad \Sigma y\,dm = M\bar{y},$$

where $\bar{x}$, $\bar{y}$ are the coordinates of the centre of mass of the lamina.

Hence the resultant of the entire system of centrifugal forces is the same as that of the two forces

$$\omega^2 M\bar{x} \quad \text{and} \quad \omega^2 M\bar{y};$$

or to the single force $\omega^2 Md$, where d denotes the distance of the centre of mass of the lamina from the fixed axis.

97. A similar theorem holds for any uniform rigid body turning round a fixed axis, provided the body has a *plane of symmetry* passing through the axis.

For the body may be conceived divided into a number of

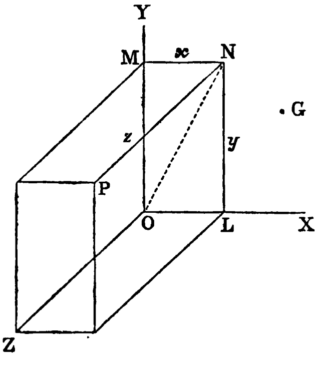
indefinitely thin parallel plates, by planes perpendicular to the fixed axis, and the preceding theorem holds for each of these plates or laminæ.

Again, if m_1, m_2, m_3, . . . &c., denote the masses of the plates; and p_1, p_2, p_3, . . . &c., the distances of their respective centres of mass from the fixed axis; then as the body is supposed uniform and symmetrical, the centres of mass of each of the plates all lie in the plane of symmetry; therefore the forces $\omega^2 m_1 p_1$, $\omega^2 m_2 p_2$, . . . &c., form a system of parallel forces. They consequently have a single resultant, equal to their sum, or to $\omega^2 Md$, where M denotes the mass of the body, and d the distance of its centre of mass from the axis of rotation.

Hence, the *stress* on the fixed axis produced by the centrifugal force is in this case represented by $\omega^2 Md$.

If the fixed axis pass through the centre of mass, the stress on the axis becomes a momental stress or a couple: as will be shown also in the following Article.

98. **Centrifugal Forces arising from Rotation for a Rigid Body.**—Next let us consider the case of any rigid body rotating round a fixed axis. Suppose a plane drawn through the centre of mass, perpendicular to the fixed axis, and meeting it in the point O. Take O as the origin, the fixed axis as that of z, and two rectangular axes as those of x and y, respectively. Let dm denote an element of mass at the point P, whose coordinates are x, y, z; then, by Art. 96, the centrifugal force for the element dm is equivalent to the forces, $\omega^2 x dm$ and $\omega^2 y dm$, acting at P, parallel to OX and OY, respectively.



Transferring these forces to the point N, they are equivalent to the forces $\omega^2 x dm$, $\omega^2 y dm$, parallel to OX and OY; along with the couples $\omega^2 xz dm$, $\omega^2 yz dm$, parallel to the planes XZ and YZ, respectively. The resultant of the forces $\omega^2 x dm$, $\omega^2 y dm$, acting at N, is obviously directed to O; and

consequently if it be transferred to O, it can be resolved into $\omega^2 xdm$ and $\omega^2 ydm$, acting along OK and OY, respectively.

If each centrifugal force be resolved in like manner, the whole system is equivalent to the forces

$$\omega^2 \Sigma xdm \quad \text{and} \quad \omega^2 \Sigma ydm,$$

or to

$$\omega^2 M\bar{x} \quad \text{and} \quad \omega^2 M\bar{y},$$

acting along OX and OY, respectively; together with the couples $\omega^2 \Sigma xzdm$, $\omega^2 \Sigma yzdm$, acting in the planes of XZ and YZ, respectively.

If the fixed axis be a principal axis relative to the point O (*Int. Calc.*, Art. 214), we have

$$\Sigma xzdm = 0, \quad \text{and} \quad \Sigma yzdm = 0.$$

Hence, in this case the strain on the axis produced by the rotation is the same as if the entire mass was concentrated at the centre of mass of the rigid body.

If, further, the fixed axis be one of the principal axes passing through the centre of mass of the body, the centrifugal forces arising from the rotation produce no strain on the fixed axis. And accordingly, if, from any cause, a rigid body commence to rotate about such an axis, it will continue to rotate permanently round the axis, provided the only external force be that of gravity.

For example, if we suppose a homogeneous sphere, whose centre is fixed, to receive any impulse, it will commence to rotate around some one of its diameters; and, as every diameter is in this case a principal axis, it follows, from the preceding, that it will continue to revolve permanently round that axis, if we suppose no external force but gravity to act on it.

On account of the property established above, it is of importance, in order that any machine should work smoothly, that the centre of mass of any wheel, or portion which rotates rapidly, should lie on the axis of rotation, which should be a principal axis; for otherwise the centrifugal forces would cause strong disturbing vibrations.

The theorems of this section are particular cases of important general results, which will be discussed in a subsequent chapter.

Examples.

1. A string of 5 feet length is just capable of supporting a weight of 10 lbs.: find the greatest number of revolutions per minute that a weight of 4 lbs. attached to the extremity of the string is capable of making in a horizontal plane without breaking the string. *Ans.* 38.

2. A mass of 8 lbs. is suspended from the extremity of a string 10 feet long: find the least velocity that should be given to it in order to break the string, if its breaking tension be 12 lbs. *Ans.* 12·64 feet per second.

3. Two balls weighing 6 lbs. each are fixed at the extremities of a rod of 10 feet length, which revolves 100 times in a minute around a central vertical axis; find the tension of the connecting rod. *Ans.* 102 lbs.

4. If two equal bodies moving on a rough horizontal plane be connected by a string of invariable length a, but without weight; find the longest time that one can continue to move after the other has been stopped by friction.

Ans. $\sqrt{\dfrac{a}{\mu g}}$, where μ is the coefficient of friction.

5. A body m sliding on a perfectly smooth horizontal table is connected by a string passing through a smooth hole in the table, with another body m' which hangs freely; find the condition that m' should remain at rest, and also the time of revolution of m in its circular path, supposed of radius a.

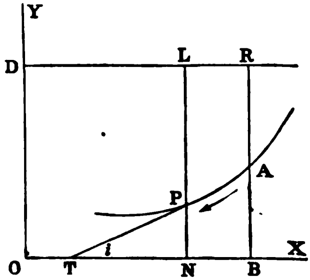
Ans. Velocity of m should be $\sqrt{\dfrac{m'ga}{m}}$: time of revolution $= 2\pi\sqrt{\dfrac{ma}{m'g}}$.

6. If a body, attached at its centre of mass to one end of a string of length r, the other end of which is attached to a fixed point on a smooth horizontal plane, makes n revolutions per second; prove that the tension of the string is to the pressure on the plane as $4\pi^2 n^2 r$ to g.

Prove that at the Equator a shot fired westward, with velocity 8333, or eastward, with velocity 7407 metres per second, will, if unresisted, move horizontally round the earth in one hour and twenty minutes, and one hour and a-half respectively. *Camb. Trip.*, 1878.

7. A rigid body of any form revolves freely round an axis fixed in space: required the conditions under which the centrifugal forces of its several elements will have—(a) no resultant; (b) a resultant pair; (c) a resultant single force; (d) a resultant pair and single force. *Lloyd Exhib.*, 1872.

Section III.—*Motion in a Vertical Circle.*



99. Velocity in a Smooth Vertical Curve.—Before the discussion of motion in a circle we shall consider some properties of the motion of a particle, under the action of gravity, on any vertical curve.

Take OX a horizontal, and OY a vertical line in the plane as axes of coordinates; and suppose x, y the coordinates of P, the position of the particle at the

end of any time t. Let A be its position when $t = 0$, and let $AP = s$; then, by Art. 43, the acceleration along the curve at P is represented by

$$g \sin i = -g \frac{dy}{ds}.$$

Hence we have
$$\frac{d^2s}{dt^2} = -g \frac{dy}{ds}. \qquad (1)$$

Multiply by $2ds$, and integrate; then

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -2gy + \text{const.}$$

Let $y_0 = AB$, and $v_0 =$ velocity at A, then

$$v_0^2 = -2gy_0 + \text{const.};$$

therefore
$$v^2 - v_0^2 = 2g(y_0 - y). \qquad (2)$$

Again, if AR be measured upwards $= h$, the height due to the velocity v_0, and RD be drawn parallel to the axis of x; then

$$v^2 = 2g(h + y_0 - y) = 2gPL. \qquad (3)$$

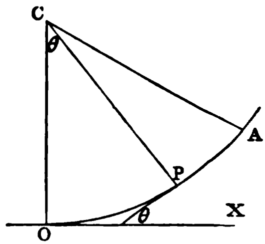
Consequently *the velocity at any point P is the same as that acquired in falling from the horizontal line DR.*

This is an extension of the result given in Art. 49, and is itself a case of the general principle of work which shall be treated of in the next chapter (*see* Art. 132).

100. **Motion in a Vertical Circle.**—If a particle be constrained to move in a vertical circle under the action of gravity, its velocity at any point, by (2), is the *velocity due to* falling through a certain *height* from a certain horizontal line, or *level.* The motion will be one of complete revolution if this right line lies altogether outside the circle. If the line cut the circle the motion will be oscillatory. We proceed to consider the latter case in the first instance. In this case we may either consider the particle as moving in a smooth circular tube, or as attached by an inextensible string to a fixed point in the centre of the circle, the weight of the string being neglected.

When the arc in which the oscillation has place is but a small portion of the circumference we get what is called a simple pendulum. From this statement the student will see that a *simple pendulum* can only be approximately represented. However, a small leaden ball suspended from a fixed point by a very fine wire may be regarded approximately as a simple pendulum.

101. **Simple Pendulum.**—Let C be the centre of the circle; O its lowest point; A the point from which the particle may be supposed to start; P its position at the end of any time t; v the corresponding velocity,



$$\theta = \angle PCO, \quad \alpha = \angle ACO,$$

estimated in circular measure,

$$s = AP, \quad l = OC.$$

Then, since the velocity at P is that due to falling from a horizontal line drawn through A, we have

$$v^2 = 2gl(\cos\theta - \cos\alpha);$$

but $$v^2 = \left(\frac{ds}{dt}\right)^2 = l^2\left(\frac{d\theta}{dt}\right)^2;$$

therefore $$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l}(\cos\theta - \cos\alpha)$$

$$= \frac{4g}{l}\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right).$$

Consequently $$\frac{d\theta}{dt} = \pm 2\sqrt{\frac{g}{l}}\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}.$$

Again, since in the motion from A to O, θ diminishes as t increases, $\frac{d\theta}{dt}$ is negative. Accordingly we have

$$\frac{d\theta}{dt} = -2\sqrt{\frac{g}{l}}\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}.$$

Hence we get $$t\sqrt{\frac{g}{l}} = -\int_{\alpha}^{\theta} \frac{d\theta}{2\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}. \tag{4}$$

102. **Time of a Small Oscillation.**—The preceding definite integral, which represents the time of describing a circular arc, cannot be expressed in finite terms by means of the ordinary algebraic or trigonometrical functions; however, when the amplitude of the oscillation is small we can easily get an approximate value for t, as follows:—When α is so small, that we may neglect powers of α and θ beyond the second, we have

$$4\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right) = \alpha^2 - \theta^2.$$

Hence (4) becomes

$$\sqrt{\frac{g}{l}}\, t = -\int_{\alpha}^{\theta} \frac{d\theta}{\sqrt{\alpha^2 - \theta^2}} = \cos^{-1}\left(\frac{\theta}{\alpha}\right).$$

No constant is added as $\theta = \alpha$ when $t = 0$. Consequently we have

$$\theta = \alpha \cos\sqrt{\frac{g}{l}}\, t. \tag{5}$$

Accordingly θ is a simple harmonic function of the time (Art. 88).

Again, when $\theta = 0$, we have $\sqrt{\frac{g}{l}}\, t = \frac{\pi}{2}$; hence the time of descent to the lowest point is represented by $\frac{\pi}{2}\sqrt{\frac{l}{g}}$.

The particle, after arriving at the lowest point, plainly moves up the other side of the arc, and if the whole time of a small oscillation, *expressed in seconds*, be denoted by T, we have

$$T = \pi\sqrt{\frac{l}{g}}. \tag{6}$$

Since this expression is independent of α, it follows that

the *time of a small oscillation is the same for all arcs of vibration in the same circle.* From this property the vibrations of a pendulum are said to be *isochronous.* Also the time of a small oscillation at any place varies as the square root of the length of the pendulum.

103. **The Seconds Pendulum.**—A pendulum which oscillates once in every second is called a seconds pendulum. If L be its length, since the corresponding value of T is unity, we have

$$g = \pi^2 L. \tag{7}$$

Hence the value of g can be determined for any place whenever the corresponding value of L is obtained.

This gives the most accurate method of finding the value of g at any place, since that of L can be determined with great accuracy by observation.

Any rigid body made to vibrate about a fixed horizontal axis is called a *compound pendulum*. It will be shown subsequently (*see* Art. 135) that in every such case there is an *equivalent simple* pendulum which would vibrate in the *same time* as the actual pendulum under consideration. This circumstance renders the consideration of the *ideal* pendulum above discussed of the utmost practical importance.

The length of a seconds pendulum at London is found to be 39·1416 inches, approximately; hence the corresponding value of g is 32·1908 feet.

Pendulum observations furnish the most accurate proof of the fact that the force of gravity acts with equal intensity on all substances, as it will be seen that the length of the simple pendulum equivalent to any compound one depends merely on the shape of the latter, but not on its material, provided it be homogeneous.

Again, if T, T' be the times of small oscillation for two pendulums of different lengths, l and l'; and if n and n' be the number of their respective vibrations in the same time (a day suppose), we shall have

$$\frac{n}{n'} = \frac{T'}{T} = \sqrt{\frac{l'}{l}}. \tag{8}$$

Hence, if the length l of any simple pendulum be known,

and also the number n of its vibrations in a day, the length L of the seconds pendulum at the place can be calculated. For, since the number of seconds in a day is 86400, we have, from formula (8),

$$L = \left(\frac{n}{86400}\right)^2 l. \qquad (9)$$

The time T of vibration of a pendulum varies either—(1) by altering the length l of the pendulum, or (2) by changing the place of vibration. We shall consider these causes independently.

104. **Change of Length.**—Adopting the same notation as before, we get

$$\frac{n^2}{n'^2} = \frac{l'}{l};$$

hence $$\frac{n^2 - n'^2}{n'^2} = \frac{l' - l}{l}; \quad \therefore \ n - n' = \frac{n'^2}{n + n'} \, \frac{l' - l}{l}.$$

When the change in length is a very small fraction of the whole length, n and n' are nearly equal, and we have, approximately,

$$\frac{n'^2}{n + n'} = \frac{n}{2}.$$

Accordingly, in this case,

$$n - n' = \frac{n}{2} \, \frac{\Delta l}{l}; \qquad (10)$$

where Δl denotes the change of length of the pendulum.

If the pendulum be lengthened, *i.e.* if Δl be positive, $n - n'$ is positive, and hence the number of vibrations in a given time is diminished when the length of the pendulum is increased, as is otherwise evident.

In the case of a seconds pendulum we substitute L for l in the preceding; and since $n = 86400$, we get for the diminution in the number of vibrations in a day,

$$43200 \, \frac{\Delta L}{L}.$$

Hence we can determine the number of seconds gained or lost by a seconds pendulum in a day when its length is slightly altered.

As bodies in general expand slightly with an increase of temperature, an ordinary clock should go slower in hot weather, and faster in cold. The different methods of compensation for correcting the error arising from this cause will be found in practical treatises on the subject. The amount of expansion for an increase of temperature for different substances has been accurately determined, and registered in Tables.

If ΔL denote the change in the length of a seconds pendulum arising from this cause, the corresponding loss or gain can be determined by (10).

We add a few examples for illustration.

Examples.

1. Calculate the length of a pendulum beating seconds in London, assuming $g = 32{\cdot}19$.

2. If the bob of a seconds pendulum be screwed up one turn, the screw being 32 threads to the inch; find the number of seconds it should gain in the day in consequence, assuming $L = 39{\cdot}14$ inches. *Ans.* 34·7 seconds.

3. A heavy ball, suspended by a fine wire, makes 885 oscillations in an hour. Find the length of the wire approximately, assuming the length of the seconds pendulum to be 39·14 inches. *Ans.* 54 feet.

4. Find the error in one day produced by an increase of 15° F. of temperature in a steel seconds pendulum; assuming that $\frac{\Delta l}{l}$ for 10° F. $= \frac{1}{15600}$. *Ans.* 4·15 seconds.

5. A seconds pendulum is lengthened $\frac{1}{10}$th of an inch; find the number of seconds it will lose in one day. *Ans.* 110·4.

105. **Change of Place.**—The acceleration g varies* from place to place, and consequently the number of vibra-

* For places at the sea level, this arises from two causes—one, the variation of centrifugal force already considered (Art. 94); the other, that the Earth is not an exact sphere, but is more nearly an oblate spheroid of revolution round its axis of rotation. From each of these it arises that the value of g diminishes in proceeding from the pole to the equator. It was from the observation by Richer, in 1672, that a clock lost two minutes daily when taken to Cayenne, lat. 5° N., and that when the corrected pendulum was brought back to Paris it gained an equal amount, that the variation of the force of gravity on the Earth's surface was first established. The explanation is due to Huygens.

tions in a given time will vary with the place for the same pendulum.

Suppose n and n' to represent the number of vibrations made in one day by the same pendulum at two places, at which g and g' are the corresponding accelerations, we have

$$\frac{n'}{n} = \frac{T}{T'} = \sqrt{\frac{g'}{g}}, \text{ or } \frac{n'^2}{n^2} = \frac{g'}{g}.$$

Hence, as before, for one and the same pendulum,

$$n' - n = \frac{n}{2}\frac{g' - g}{g}. \tag{11}$$

From this, if L and L' be the lengths of the seconds pendulum at the two places, we get

$$n' - n = \frac{n}{2}\frac{L' - L}{L}. \tag{12}$$

It is shown by theory, and verified by observation, that the variation in the length of L, and consequently in g, at the sea level, is proportional to the square of the sine of the latitude (compare Art. 94). Thus, if L denote the length of the seconds pendulum at the equator, L' that at latitude λ', we have

$$L' = L + m \sin^2 \lambda'. \tag{13}$$

Hence, if L_1 be the length of the seconds pendulum at 45° latitude, we have $L_1 = L + \frac{m}{2}$.

Eliminating L, we get

$$L' = L_1 - \frac{m}{2} \cos 2\lambda'. \tag{14}$$

Again, if L'' be the length corresponding to the latitude λ'', and g'' the corresponding value of g, we have

$$\frac{g' - g''}{g'} = \frac{m}{2L'} (\cos 2\lambda'' - \cos 2\lambda')$$

$$= \frac{m}{L_1} \sin (\lambda' + \lambda'') \sin (\lambda' - \lambda''), \text{ approximately.}$$

By accurate observation of the number of vibrations lost by a pendulum which beats seconds at the latitude λ, when taken to a latitude λ', the value of $\frac{m}{L_1}$ can be determined.

Such observations give $\frac{m}{L_1} = \frac{1}{195}$, approximately, and $L_1 = 39{\cdot}118$ inches.

Hence, we get

$$L' = 39{\cdot}118 - \tfrac{1}{10}\cos 2\lambda'.$$

Again, suppose a pendulum, beating seconds at any place, taken to the height h above the Earth's surface at that place; and let g' be the value of g for the new position; then, since the force of gravity varies as the inverse square of the distance from the Earth's centre, we have

$$g' = g\frac{r^2}{(r+h)^2} = g\left(1 - \frac{2h}{r}\right), \text{ approximately,}$$

where r denotes the length of the Earth's radius; therefore

$$\frac{g'-g}{g} = -\frac{2h}{r}.$$

Hence, when $\frac{h}{r}$ is a very small fraction, the number of seconds lost in a day by the seconds pendulum is $86400\,\frac{h}{r}$.

Suppose, for example, $h = 1$ mile, and $r = 3956$ miles, then the number of seconds lost in a day will be 22, approximately.

In this investigation the attraction on the pendulum of the part of the Earth above the sea level has been neglected.

Examples.

1. If a pendulum, beating seconds at the foot of a mountain, lose 10 seconds in a day when taken to its summit; find approximately the height of the mountain, assuming the radius of the Earth 4000 miles, and neglecting the attraction of the mountain. *Ans.* 2444 feet.

2. How much would a clock gain at the equator in 24 hours if the length of the day were doubled. *Ans.* $112\frac{1}{2}$ seconds, approximately.

106. **Airy's Investigation of the Mean Density of the Earth.**—A series of important pendulum experiments were undertaken by Sir G. B. Airy, in the Harton coal mine, for the purpose of determining the mean density of the Earth. He found that a pendulum beating seconds at the surface gained $2\frac{1}{4}$ seconds a-day when taken to the bottom of the mine, 1260 feet deep. The calculations employed in arriving at this result, and in determining from it the Earth's mean density, are very intricate; they will be found in the Royal Society's *Transactions* for the year 1856.

The following is a method of arriving, approximately, at the result:—

Let g, g' denote the accelerations due to gravity at the surface and at the bottom of the mine; then, by equation (11), we have

$$\frac{g'-g}{g} = \frac{2\frac{1}{4}}{43200} = \frac{1}{19200}.$$

Again, let r and r' denote the distances of the upper and lower stations from the centre of the Earth, supposed spherical.

Suppose a concentric sphere described through the lower station, then the attraction of the Earth at the upper station may be regarded as consisting of two parts—one due to the interior sphere, the other to the *conche* or shell, bounded by the two spheres. Again, if we suppose this shell to be of uniform density, it exercises no attraction on the pendulum at the bottom of the mine. This can be easily seen from elementary geometrical considerations (Minchin, *Statics*, Art. 319). Hence the part of g due to the attraction of the *inner* sphere is represented by

$$g' \frac{r'^2}{r^2}.$$

If f denote the acceleration at the upper station due to the attraction of the shell, we have

$$g = f + g' \frac{r'^2}{r^2} = f + g\left(1 + \frac{1}{19200}\right)\frac{r'^2}{r^2}.$$

Again, let h represent the depth of the mine, then $r' = r - h$;

and, since $\frac{h}{r}$ is very small, we have $\frac{r'^2}{r^2} = 1 - \frac{2h}{r}$, approximately.

Accordingly we get, from the preceding equation,

$$f = g\left(\frac{2h}{r} - \frac{1}{19200}\right).$$

In order to get another expression for f, let M, V, D denote respectively the mass, volume, and mean density of the Earth; and m, v, ρ the corresponding quantities for the shell. We assume that the Earth and the shell each attract as if their whole mass was concentrated at their common centre; in this case we have, approximately,

$$f = g\frac{m}{M} = g\frac{\rho}{D}\frac{v}{V} = g\frac{\rho}{D}\frac{r^3 - r'^3}{r^3} = 3g\frac{\rho}{D}\frac{h}{r}.$$

Hence
$$\frac{2h}{r} - \frac{1}{19200} = 3\frac{h}{r}\frac{\rho}{D}.$$

Substituting 3956 miles for r, and 1260 feet for h, we get

$$D = 2{\cdot}625\rho. \tag{15}$$

The determination of the mean density of the Earth is thus reduced to finding the value of ρ; but this is a matter of extreme practical difficulty.

From an accurate examination of the mineral components of the stratum of the Earth in the neighbourhood of the mine, ρ was calculated by Airy to be $2\frac{1}{2}$ times the density of water. This would give the mean density of the Earth about 6·66, assuming that of water as unity.

Professor Haughton calculated 2·059 as the value of ρ (*Phil. Trans.*, July, 1856), adopting as his basis Humboldt's investigations of the mean heights of Continents on the Earth's surface, and Rigault's, on the relative areas of land and water. This would give 5·405 as the value of the mean density of the Earth.

107. **Time of Oscillation in General.**—The amplitude of the vibration has hitherto been considered so small that powers of a higher than the second have been neglected.

We now proceed to find a general expression for the time T of vibrations for any amplitude.

From (4), since $\frac{1}{4}T$ represents the time to the lowest point on the circle, we get

$$T = \sqrt{\frac{l}{g}}\int_0^\alpha \frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}. \tag{16}$$

Now, assuming* $\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\sin\phi$, we get

$$\frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}} = \frac{2d\phi}{\cos\frac{\theta}{2}} = \frac{2d\phi}{\sqrt{1 - \sin^2\frac{\alpha}{2}\sin^2\phi}}.$$

Also, when $\theta = 0$ we have $\phi = 0$; and when $\theta = \alpha$ we have $\phi = \frac{\pi}{2}$.

Consequently $$T = 2\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \sin^2\frac{\alpha}{2}\sin^2\phi}}. \tag{17}$$

Again, substitute k^2 for $\sin^2\frac{\alpha}{2}$; then, since

$$(1 - k^2\sin^2\phi)^{-\frac{1}{2}} = 1 + \frac{1}{2}k^2\sin^2\phi + \frac{1 . 3}{2 . 4}k^4\sin^4\phi + \frac{1 . 3 . 5}{2 . 4 . 6}k^6\sin^6\phi + \&c.,$$

and (*Int. Calc.*, Art. 93),

$$\int_0^{\frac{\pi}{2}} \sin^{2m}\phi\, d\phi = \frac{1 . 3 . 5 \ldots (2m-1)}{2 . 4 . 6 \ldots\ 2m}\,\frac{\pi}{2},$$

we get

$$T = \pi\sqrt{\frac{l}{g}}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 . 3}{2 . 4}\right)^2 k^4 + \left(\frac{1 . 3 . 5}{2 . 4 . 6}\right)^2 k^6 + \&c.\right\} \tag{18}$$

---

* This assumption is obviously a legitimate one, because θ during the motion can never be greater than α.

If h be the vertical height of the point from which the pendulum commences to descend above the lowest point in the circle, we have

$$k^2 = \sin^2\frac{a}{2} = \frac{1-\cos a}{2} = \frac{h}{2l};$$

and the preceding result becomes

$$T = \pi\sqrt{\frac{l}{g}}\left\{1 + \left(\frac{1}{2}\right)^2\frac{h}{2l} + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2\left(\frac{h}{2l}\right)^2 + \left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2\left(\frac{h}{2l}\right)^3 + \ldots\right\} \quad (19)$$

The first term gives the value already arrived at for a small oscillation, and is independent of the amplitude.

The second approximation, which is the one commonly adopted when the lowest powers of the amplitude are taken into account, gives

$$T = \pi\sqrt{\frac{l}{g}}\left(1 + \frac{h}{8l}\right). \quad (20)$$

In terms of the semi-amplitude a, this is

$$T = \pi\sqrt{\frac{l}{g}}\left\{1 + \frac{a^2}{16}\right\}, \quad (21)$$

in which a is taken in circular measure.

The general equation (16) is immediately integrable in one case, viz., when the velocity at any point on the circle is that due to a fall from its highest point; for in this case we have $a = \pi$, and therefore $\sin\frac{a}{2} = 1$. Equation (4) becomes in this case

$$\frac{d\theta}{\cos\frac{1}{2}\theta} = 2\sqrt{\frac{g}{l}}\,dt;$$

hence we get

$$2\sqrt{\frac{g}{l}}\,t = \log\tan\tfrac{1}{4}(\pi + \theta) + \text{const.}$$

It may be observed that in this case, since $\log 0 = -\infty$, the particle would take an infinite time to reach the highest point on the circle.

EXAMPLES.

1. How is the value for the time of vibration of a pendulum to be corrected when the length of the arc of vibration is taken into account?

2. Apply to the case where the amplitude of vibratiou is 120°.

Here, since $\frac{h}{2l} = \frac{1}{4}$, we have

$$T = \pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16} + \frac{9}{1024} + \&c.\right)$$

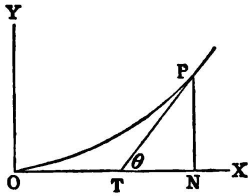
3. If a pendulum, which beats seconds for very small oscillations, be made to vibrate through an arc of 10°; find, approximately, the number of seconds it should lose in a day. *Ans.* 41.

108. **Motion in a Vertical Cycloid.**—Let a particle be supposed to move along a smooth cycloid, having its vertex O at its lowest point, and its axis OY vertical.

Calling $OP = s$, $PN = y$, and a = diameter of generating circle.

Then (*Diff. Calc.*, Art. 276), we have

$$s^2 = 4ay.$$



Also, from (1),

$$\frac{d^2s}{dt^2} = -g\frac{dy}{ds} = -\frac{g}{2a}s,$$

or

$$\frac{d^2s}{dt^2} + \frac{g}{2a}s = 0. \tag{22}$$

We shall next consider the method of integrating this equation.

109. **Integration of** $\frac{d^2x}{dt^2} \pm \mu x = 0$.—As differential equations of the form $\frac{d^2x}{dt^2} \pm \mu x = 0$ are of frequent occurrence in physical problems, we proceed to consider their solution.

There are two cases, according as the upper or lower sign is taken.

1st.—Let $$\frac{d^2x}{dt^2} + \mu x = 0.$$

Multiplying by $2dx$, and integrating, we get

$$\left(\frac{dx}{dt}\right)^2 + \mu x^2 = \text{const.}$$

To determine the constant, suppose $x = a$ when $\frac{dx}{dt} = 0$, then the constant is, plainly, μa^2;

therefore $$\left(\frac{dx}{dt}\right)^2 = \mu (a^2 - x^2).$$

Hence $$\frac{dx}{\sqrt{a^2 - x^2}} = \sqrt{\mu} dt,$$

or $$\sin^{-1}\frac{x}{a} = t\sqrt{\mu} + \alpha,$$

where α denotes an arbitrary constant.

Consequently $$x = a \sin (t\sqrt{\mu} + \alpha), \tag{23}$$

where a, α are arbitrary constants, to be determined in each case by the conditions of the problem.

It may be observed that x is a simple harmonic function of the time (Art. 88).

The preceding solution admits of being also written in the form

$$x = C \cos t\sqrt{\mu} + C' \sin t\sqrt{\mu}, \tag{24}$$

where C and C' are two arbitrary constants.

Either of the latter equations may be regarded as the complete integral of the differential equation

$$\frac{d^2x}{dt^2} + \mu x = 0.$$

2nd.— $$\frac{d^2x}{dt^2} = \mu x.$$

Proceeding as before, we get

$$\left(\frac{dx}{dt}\right)^2 = \mu(x^2 - a^2),$$

in which a is an arbitrary constant;

or $$\frac{dx}{\sqrt{x^2-a^2}} = dt\sqrt{\mu};$$

therefore, $$t\sqrt{\mu} + \alpha = \int\frac{dx}{\sqrt{x^2-a^2}} = \log\left(x + \sqrt{x^2-a^2}\right),$$

in which α is arbitrary.

Hence $$x + \sqrt{x^2-a^2} = e^{\alpha + t\sqrt{\mu}} = Ae^{t\sqrt{\mu}},$$

where A is arbitrary.

Again, since

$$\left(x + \sqrt{x^2-a^2}\right)\left(x - \sqrt{x^2-a^2}\right) = a^2,$$

we get $$x - \sqrt{x^2-a^2} = \frac{a^2}{A}e^{-t\sqrt{\mu}}.$$

Adding, we obtain

$$2x = Ae^{t\sqrt{\mu}} + \frac{a^2}{A}e^{-t\sqrt{\mu}},$$

which may be written in the form

$$x = Ce^{t\sqrt{\mu}} + C'e^{-t\sqrt{\mu}}, \tag{25}$$

when C and C' are two arbitrary constants, to be determined, as before, by the conditions of the problem in each particular case.

110. The equation

$$\frac{d^2x}{dt^2} + \mu x + \nu = 0$$

is immediately reducible to the preceding, for it may be written

$$\frac{d^2x}{dt^2} + \mu\left(x + \frac{\nu}{\mu}\right) = 0.$$

If we substitute z for $x + \frac{\nu}{\mu}$, this becomes $\frac{d^2z}{dx^2} + \mu z = 0$; consequently we have

$$x = -\frac{\nu}{\mu} + C \cos t\sqrt{\mu} + C' \sin t\sqrt{\mu}.$$

In like manner the solution of

$$\frac{d^2x}{dt^2} - \mu x + \nu = 0$$

is
$$x = \frac{\nu}{\mu} + Ce^{t\sqrt{\mu}} + C'e^{-t\sqrt{\mu}}.$$

111. **Time of Oscillation in Cycloid.**—Returning to equation (22), Art. 108, and substituting s for x, and $\frac{g}{2a}$ for μ in equation (24), we find for its integral

$$s = c \cos t\sqrt{\frac{g}{2a}} + c' \sin t\sqrt{\frac{g}{2a}}. \qquad (26)$$

In order to determine the constants c and c', suppose the particle to start from rest, at the distance s' from the vertex O (measured along the curve); then we have $s = s'$ and $\frac{ds}{dt} = 0$, when $t = 0$. Making these substitutions in (26), as well as in the equation derived from it by differentiation, we get.

$$c = s', \quad \text{and} \quad c' = 0;$$

therefore
$$s = s' \cos t\sqrt{\frac{g}{2a}}. \qquad (27)$$

Again, when $s = 0$, we get

$$t\sqrt{\frac{g}{2a}} = \frac{\pi}{2}, \text{ or } t = \frac{\pi}{2}\sqrt{\frac{2a}{g}}:$$

this gives the time of descent to the lowest point. If T denote the time of an oscillation, we have

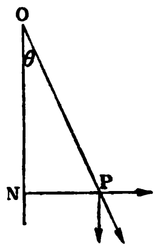
$$T = \pi\sqrt{\frac{2a}{g}}. \qquad (28)$$

Since this result is independent of the length of the arc of vibration, it follows that the *time of vibration is the same for all arcs of the cycloid;* accordingly the property of *tautochronism*, which in the circle holds only for very small arcs, holds in all cases for the cycloid (compare Art. 88).

The foregoing value of T is the same as that for a small oscillation in a vertical circle of radius $2a$. Moreover, as $2a$ is the radius of curvature at the vertex of the cycloid (*Diff. Calc.*, Art. 276), the duration of an oscillation in a vertical cycloid is the same as that of a small oscillation in the circle which osculates it at its lowest point; as is manifest also from other considerations.

It is readily seen that the time of an indefinitely small oscillation about the lowest point in any plane vertical curve is the same as that in the osculating circle at the lowest point; and its duration is accordingly represented by $\pi\sqrt{\frac{\rho}{g}}$, where ρ denotes the radius of curvature at the point.

112. **Conical Pendulum.**—Suppose the pendulum, instead of moving in a vertical plane, to describe a right cone around a vertical axis; and let P be the position of the revolving particle at any instant; O the point of suspension; PN the perpendicular let fall on the vertical axis.



Also let $\qquad OP = l, \quad \angle PON = \theta.$

Then the motion of P may be considered as taking place in a horizontal circle, whose centre is N, and radius PN or $l \sin\theta$.

Now, in order that this motion should take place, it is necessary that the resultant of the tension of the string and the weight of the particle should act along PN, and be equal and opposite to the centrifugal force; *i.e.* that the resultant of the weight W, and the centrifugal force, $\frac{W}{g}\frac{v^2}{l\sin\theta}$, should act in the line OP. This gives

$$W : \frac{W}{g}\frac{v^2}{l\sin\theta} = ON : PN,$$

or $$v^2 = gl\sin\theta\,\frac{PN}{ON} = gl\frac{\sin^2\theta}{\cos\theta};$$

therefore $$v = \sin\theta\sqrt{\frac{gl}{\cos\theta}}. \qquad (29)$$

This gives the velocity in terms of θ and l.

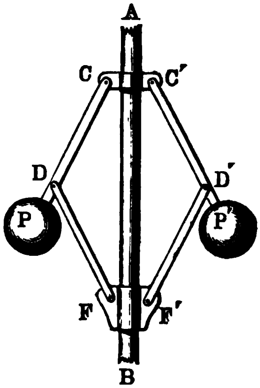
Again, if T be the time of revolution, we have

$$T = \frac{2\pi PN}{v} = 2\pi\sqrt{\frac{l\cos\theta}{g}}. \qquad (30)$$

This determines the time of revolution when the angle θ, which the pendulum makes during the motion with the vertical, is known. It is evidently the same as that of a double oscillation in a simple pendulum of length $l\cos\theta$ or ON. The tension of the string is represented by $W\sec\theta$. The preceding is a particular case of the motion of a particle on a smooth sphere, a problem which will be considered in Chapter VIII..

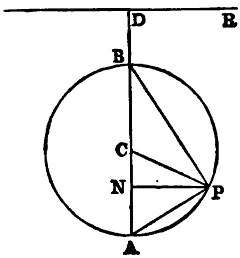
113. **Watt's Governor.**—The principal of the conical pendulum was employed by Watt, in the instrument called a governor, for the purpose of regulating the supply of steam so as to maintain, approximately, a steady motion in a steam-engine. Its construction, under a form which is commonly employed, is as follows :—

Let AB represent a vertical spindle rotating with an angular velocity, whose speed is so regulated as to be always proportional to that of the machine. CP and $C'P'$ are rigid rods, jointed at C and C' upon the revolving spindle, and having massive equal balls, P and P', fixed at their extremities. FD and $F'D'$ are two rods also jointed at D and D' to the rigid rods, and jointed at F and F' to a *collar*, movable freely on the spindle. The collar at F, sliding freely up and down the spindle, is united to a lever which opens or closes the valve that regulates the supply of steam to the cylinder of the engine. When the shaft AB turns too fast, the balls P and P' fly from it, raising the collar F, and thus diminishing the supply of steam, and consequently reducing the speed. For a more complete discussion the student is referred to works on practical mechanics.



114. Revolution in a Vertical Circle.—We now return to the question of the revolution of a particle in a vertical circle under the action of gravity.

Suppose DR to be the horizontal line to the distance below which the velocity at any point is due, and let $AD = h$; then, by Art. 99, the velocity at any point P is given by the equation



$$v^2 = 2g(h - AN) = 2g(h - 2a\sin^2\tfrac{1}{2}\theta),$$

where $PCA = \theta$.

Hence, denoting $\dfrac{2a}{h}$ by k^2, and substituting $a^2\left(\dfrac{d\theta}{dt}\right)^2$ for v^2, we get

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{4g}{ak^2}(1 - k^2\sin^2\tfrac{1}{2}\theta);$$

therefore $$\frac{d\theta}{dt} = -\frac{2}{k}\sqrt{\frac{g}{a}}\sqrt{1-k^2\sin^2\tfrac{1}{2}\theta},$$
in which k is less than unity.

If $\phi = \angle PBA = \frac{1}{2}\theta$, the time of describing any arc of the circle is represented by the definite integral

$$k\sqrt{\frac{a}{g}}\int_\beta^\alpha \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \tag{31}$$

where α and β are the values of ϕ corresponding to the extremities of the arc.

Comparing the result here given with Art. 107, we see that the time of describing any arc of a circle is in this case in a constant ratio to the time of describing a corresponding arc of a second circle, in which the motion is oscillatory.

The time of describing any arc of the circle is, in general, an elliptic function. There is one case, however, in which it admits of a simple expression, viz., where DR is a tangent to the circle, as in Art. 107.

In this case we have $k = 1$, and the definite integral becomes

$$\int_\beta^\alpha \frac{d\phi}{\cos\phi} = \log\frac{\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}{\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right)}.$$

The time of motion from any point to the highest point in the circle becomes infinite in this case, as already observed in Art. 107; accordingly the particle would continually approach the highest point without ever reaching it.

115. **Pressure on Curve.**—If m denote the mass of the particle, then the normal pressure R on the circle consists of two parts—one arising from the centrifugal pressure, the other from the weight—hence we get

$$R = m\frac{v^2}{a} + mg\cos\theta$$

$$= mg\left(\frac{2d}{a} + 3\cos\theta\right), \tag{32}$$

where $CD = d$.

At the lowest point this becomes $mg\left(\frac{2d}{a}+3\right)$, at the highest point, $mg\left(\frac{2d}{a}-3\right)$; and when the string is horizontal, $mg\frac{2d}{a}$.

If the particle, instead of moving in a tube, is attached by a string, of length a, to a fixed point C, and thus constrained to move in a vertical circle, the preceding expression gives the tension of the string for any position. As long as the tension is positive the string remains stretched. At the point where $R = 0$ the tension vanishes, and the particle will leave the circle and proceed to describe a parabola. It is immediately seen that the distance of this point from the line DR is one-third of CD (see fig. of last Article).

These results will be illustrated by the following examples:—

Examples.

1. A particle slides down the convex side of a vertical circle; determine the point at which it will leave the curve.

Here, since the velocity at the highest point on the circle is zero, we have $d = 0$; accordingly the point at which $R = 0$ is given by the equation $\cos\theta = -\frac{2}{3}$. The geometrical construction is evident.

2. A particle is projected from the lowest point along the inside of a smooth vertical circle; find the least velocity of projection in order that the particle should make a complete revolution in the circle. *Ans.* $\sqrt{5ag}$.

In this case the pressure at the highest point is zero, and at every other point is positive.

3. If the initial velocity be less than that in the preceding example, find the point P at which the particle will leave the circle, and where it will strike it again.

The construction for the point P in question has been given above. After leaving the circle the particle describes a parabola, and the point Q in which it again meets the circle is found by drawing PQ, making with the vertical direction an angle equal to that which the tangent at P makes with the vertical. This result follows immediately from the principle that the vertical circle osculates the parabolic trajectory at P.

4. In the same case find the direction of motion of the particle at the instant it returns to the circle.

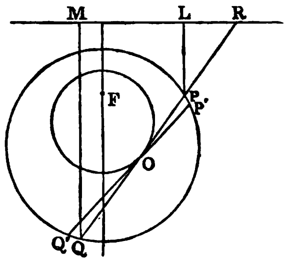
Ans. $\tan\beta = \frac{1}{3}\tan\alpha$, where β is the angle which the required direction of motion makes with the vertical; and α is the corresponding angle at the point P, where the particle leaves the circle.

5. Find the velocity of projection from the lowest point on the circle, in order that the particle after leaving the circle should meet it again at its lowest point. *Ans.* $\frac{1}{2}\sqrt{14ga}$.

6. Show that the solution of the general problem of finding the initial velocity, in order that the particle after leaving the circle shall meet it again at a given point, depends on the trisection of an arc.

7. A material particle moves in a circular groove on a smooth inclined plane; if it be projected from its point of rest with a velocity just sufficient to carry it to the highest point in the groove, find the time of its motion.

116. **Lemma on Coaxal Circles.**—A chord PQ of a circle touches a second circle at O; and PL, QM are drawn perpendicular to the radical axis of the two circles: to prove that



$$PO^2 : QO^2 = PL : QM.$$

Let R be the point of intersection of PQ with the radical axis; then, since the tangents from R to the circles are of equal length, we have $RO^2 = RP \cdot RQ$;

therefore $\quad RQ : RO = RO : RP.$

Consequently $\quad QO : OP = RQ : RO,$

or

$$QO^2 : OP^2 = RQ^2 : RQ \cdot RP$$

$$= RQ : RP = QM : PL. \qquad (33)$$

If now we suppose a particle to describe the outer circle with a velocity due to the level LM, and $P'Q'$ be drawn indefinitely near to PQ, touching the inner circle, these tangents may be regarded as intersecting in O, and we accordingly have

$$PP' : QQ' = PO : QO = \sqrt{PL} : \sqrt{QM}.$$

Again, let v, v' be the velocities of the particle when at P and Q respectively; then $v^2 = 2g\,PL$, $v'^2 = 2g\,QM$;

therefore $\quad v : v' = \sqrt{PL} : \sqrt{QM} = PP' : QQ'.$

Hence the time of describing PP' is the same as that of describing QQ'; consequently the time of motion from P to Q is the same as that from P' to Q', and hence we readily infer that the time is the same for the description of all arcs cut off by tangents drawn to the inner circle.

EXAMPLES.

1. A particle is moving in a smooth vertical circle under the action of gravity: the time of description of a variable arc of the circle being supposed constant, show that the envelope of its chord is another circle.

2. Show that if the time of motion from P to Q be the same as that from the highest to the lowest point on the circle, the line PQ always passes through a fixed point.

3. Two particles are projected from the same point, in the same direction, and with the same velocity, but at different instants, in a smooth circular tube, of small bore, whose plane is vertical. Prove that the line joining them always touches a circle.

4. In the same case, if the particles be projected in opposite directions, the other circumstances being unaltered, prove that the line joining their positions always touches a circle; and find when the circle becomes a fixed point.

5. A particle is moving in a vertical circle under the action of gravity. If three points L, M, N be taken on the circle, find a fourth point P, such that the time of motion from N to P shall be equal to that from L to M.

6. In the same case find P, so that the time of describing NP shall be double, or any given multiple of that of describing LM.

7. AB is the vertical diameter of a fine circular tube in which move three equal particles P, Q, Q' (modulus of restitution = 1 for any pair); P starts from A, and Q, Q', in opposite senses from B with such velocities that at the first impact all three have equal velocities; prove that throughout the motion the line joining any pair is either horizontal or passes through one of two fixed points, and that the intervals of time between successive impacts are all equal.

Camb. Trip., 1874.

117. **Application to Elliptic Functions.**—Since the time of description, under the action of gravity, of any arc of a vertical circle, is expressible by a definite integral of the form

$$\int_\alpha^\beta \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}},$$

the results of the last Article have important applications in the theory of elliptic functions. For example, they furnish us with simple methods for the addition, subtraction, and multiplication of such functions, depending on elementary properties of coaxal circles. This connexion was first pointed out by Jacobi (*Crelle's Journal*, 1828; *Liouville's Journal*, 1845).

Examples.

1. Prove that the time of descending any small arc terminated at the lowest point of a vertical circle is to the time down its chord as $\pi : 4$.

2. If the length of a seconds pendulum be 39·14 inches; find the corresponding value of g to two places of decimals.

3. A clock loses 4 minutes in a day; find how much its pendulum should be shortened in order that it may keep correct time. *Ans.* Its $\frac{1}{180}$th part.

4. Assuming the length L in inches of a seconds pendulum at the latitude λ to be given by the formula

$$L = 39\cdot118 - \tfrac{1}{10}\cos 2\lambda;$$

find the ratio of the difference between the values of polar and equatorial gravity to equatorial gravity. *Ans.* $\frac{1}{196\cdot09}$.

5. Find the correction in the time of vibration of a circular pendulum when the amplitude of the vibration is 30°.

6. If two particles be connected by an inextensible string, and if one be made to move as if under the action of a constant force; prove that the relative motion of the other is that of a simple pendulum.

7. A series of smooth circles in a vertical plane have a common highest point; a particle starting at this point slides down the convex side of each circle; find the locus of the point where the particles leave the circles.

8. A mass m, after sliding down the inner surface of a smooth hemispherical bowl, strikes a mass m' placed at the lowest point of the bowl. If both bodies be perfectly elastic, find the heights to which they respectively ascend after collision.

9. If the length of a conical pendulum be 1 foot, and the weight attached to its extremity be 1 lb.; find approximately the tension of the connecting wire when the time of its revolution is one second. Find also approximately the angle which the revolving wire makes with the vertical spindle.

Ans. Tension $= \dfrac{4\pi^2}{g}$ lb.; $\cos\theta = \dfrac{g}{4\pi^2}$.

10. Investigate the motion of a cycloidal pendulum when acted on by a constant force f, always in a direction opposite to that of its motion, in addition to the force of gravity.

Here the equation of motion is $$\frac{d^2s}{dt^2} + \frac{g}{2a}s = f,$$

and we get, by Art. 110, $$s = \frac{2af}{g} + c\cos\sqrt{\frac{g}{2a}}\,t + c'\sin\sqrt{\frac{g}{2a}}\,t.$$

If, when $t = 0$, $s = s'$ and $\frac{ds}{dt} = 0$, we get

$$s = \frac{2af}{g} + \left(s' - \frac{2af}{g}\right)\cos\sqrt{\frac{g}{2a}}\,t, \text{ and } \frac{ds}{dt} = -\left(s' - \frac{2af}{g}\right)\sqrt{\frac{g}{2a}}\sin\sqrt{\frac{g}{2a}}\,t.$$

This vanishes when $\sqrt{\frac{g}{2a}}\,t = \pi$; accordingly the time of an oscillation is $\pi\sqrt{\frac{2a}{g}}$; the same as when unresisted.

11. A heavy particle is connected by an inextensible string, 3 feet long, to a fixed point, and describes a circle in a vertical plane about that point, making 600 revolutions per minute; find, approximately, the ratios of the tensions of the string when the particle is at the highest and lowest points, and when the string is horizontal.

12. A body hangs freely from a fixed point by an inextensible string 2 feet in length. It is projected in a horizontal direction with a velocity of 20 feet per second. Compare the tensions at the highest and lowest points of the circle which is described, assuming $g = 32$. *Ans.* 29 : 5.

13. Show that the time of a small oscillation of a pendulum which vibrates in the air is unaffected by its resistance.

The resistance is usually assumed to vary as the square of the velocity. It can accordingly be expressed by a term of the form $\mu\left(\frac{d\theta}{dt}\right)^2$, where μ is a very small fraction; hence in this case the equation of motion may be written

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = \mu\left(\frac{d\theta}{dt}\right)^2.$$

Since μ is small, as also $\frac{d\theta}{dt}$, we get as a first approximation $\theta = a\cos\sqrt{\frac{g}{l}}\,t$, as before. If this value be substituted in $\mu\left(\frac{d\theta}{dt}\right)^2$, in accordance with the method of successive approximations, the differential equation becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = \mu a^2\frac{g}{l}\sin^2\left(t\sqrt{\frac{g}{l}}\right) = \tfrac{1}{2}\mu\frac{g}{l}a^2\left(1 - \cos 2t\sqrt{\frac{g}{l}}\right),$$

or

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}(\theta - \tfrac{1}{2}\mu a^2) = -\mu\frac{g}{2l}a^2\cos 2t\sqrt{\frac{g}{l}}.$$

The integral of this, subject to the condition that $\theta = a$, and $\frac{d\theta}{dt} = 0$, when $t = 0$, is

$$\theta = \tfrac{1}{2}\mu a^2 + (a - \tfrac{2}{3}\mu a^2)\cos t\sqrt{\frac{g}{l}} + \tfrac{1}{6}a^2\mu\cos 2t\sqrt{\frac{g}{l}}.$$

Also
$$\frac{d\theta}{dt} = -\sqrt{\frac{g}{l}} \sin t\sqrt{\frac{g}{l}}\left(\alpha - \tfrac{2}{3}\mu\alpha^2 + \tfrac{1}{3}\mu\alpha^2 \cos t\sqrt{\frac{g}{l}}\right).$$

Hence, since $\frac{d\theta}{dt} = 0$ at the end of one vibration, if T be the corresponding value of t, we have $\sin T\sqrt{\frac{g}{l}} = 0$, or $T = \pi\sqrt{\frac{l}{g}}$. Accordingly, the duration of the oscillation is not affected by the resistance. Also, since we have in this case, $\cos t\sqrt{\frac{g}{l}} = -1$, the corresponding value of θ is $-(\alpha - \frac{4}{3}\mu\alpha^2)$; accordingly the resistance of the air reduces the amplitude of the oscillation by $\frac{4}{3}\mu\alpha^2$. The successive angles of oscillation diminish according to the same law, but the time of oscillation remains the same for each.

CHAPTER VI.

WORK AND ENERGY.

118. **Work.**—In all cases where force is employed in overcoming resistance so as to produce motion, work is said to be performed. Hence the conception of work involves both motion and resistance; and therefore a corresponding effort or force to overcome the resistance. In general, work may be defined as the act of producing a change in the configuration of a system in opposition to forces which resist that change. We proceed to consider how the amount of work performed in any case is to be estimated.

119. **Measure of Work.**—The simplest idea of work is derived from raising a weight through a vertical height; in which case the attracting force of the Earth is the resistance overcome. The amount of work in such cases evidently increases in proportion to the weight of the body raised and to the height to which it is raised. For example, the work done in raising one ton through a height of 10 feet is ten times that of raising it one foot, or twenty times that of raising one cwt. through 10 feet; and so on in all cases. Hence it is readily seen that the work performed in such cases is measured by the *product of the weight into the height*, i.e. by Wh, where W represents the number of units in the weight, and h that in the height.

In general, if we confine our attention to a single point which is moved in direct opposition to a constant resisting force, the work done is estimated by the product of the force and the distance through which the point is moved, *i.e.* by Pp, where P represents the force, which overcomes the equal and opposite resisting force, and p the distance passed over.

120. **Gravitation Unit of Work.**—From the ordinary units adopted in this country we derive the unit of work called a *foot-pound*, i.e. the work performed in raising

one pound through one foot in height. This is the unit usually adopted in practical local application of work, and is called the Gravitation Unit of Work (Art. 65). The corresponding unit in the metric system is called the *kilogrammetre*, or kgm. That is the work of raising a kilogramme through the height of a metre. A kilogrammetre is 7·233 foot-pounds. The unit of work in this system varies slightly from place to place with the value of g, and this should be remembered if numerical or scientific accuracy were required (Art. 39).

121. **Absolute Unit of Work.**—In the absolute system the unit of resistance is that already adopted (Art. 64) as the unit of force. Thus, if we take a poundal as the unit of force, the corresponding unit of work is that done by a poundal acting through a foot. This is sometimes called the foot-poundal. It is obvious that a foot-pound is g times a foot-poundal: accordingly, any result in the former system is reducible to the latter at any place by multiplying by the corresponding value of g.

Again, adopting the definition of a dyne given in Art. 64, the *work done by a dyne in working through a centimetre*, is called an *erg;* and a foot-poundal is 421,394 ergs.

In such measurements as are required in electrical and magnetic investigations, the absolute unit of work is always adopted, and the *erg* is the unit usually employed.

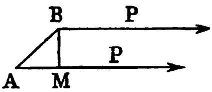
122. **Horse-power.**—Although in our definition of work we have taken no account of the time occupied in its performance, yet time becomes a necessary element when we come to compare the *efficiency* of different agents. For instance, if one agent working uniformly performs an amount of work in one hour which it requires another 5 hours to accomplish, the former is said to be five times as efficient. In comparing the work done by a steam-engine or other agent we usually adopt as our unit the horse-power defined by Watt.

Thus an engine is said to be of one-horse-power when it is capable of performing 33,000 foot-pounds of work in one minute of time, or 550 foot-pounds in one second, and so on in proportion.

Continental writers employ horse-power as 75 kgm., that is, 542·475 foot-pounds, per second.

123. Again, the work performed in raising a body of weight W to any height h is the same whether the body be raised vertically up or brought up by any other course. The whole work is still represented by Wh, where h is the space through which the weight has been moved, estimated in the vertical direction, *i. e.* in that in which the resistance of gravity acts. And, generally, the work done by any uniform effort or force, acting in a constant direction against an equal and opposite force P, is measured by the product of the force into the space through which its point of application is moved, estimated in the direction in which the force acts.

Thus, if a force P be supposed to act at A, and to move its point of application to B; then if BM be drawn perpendicular to AP, the work done is estimated by Pp, or by $P\Delta s \, . \cos\theta$, where $p = AM$, $\Delta s = AB$, and $\theta = \angle BAM$.



The work done is, therefore, regarded as positive or negative according as the angle θ, which the direction of the force makes with that of the motion, is acute or obtuse.

If $\theta = \frac{1}{2}\pi$, the direction of the motion is perpendicular to that of the force, and the work done is zero.

If two or more forces act on a system, the whole work done is the sum of the works done by each force separately.

If any number of forces be in equilibrium, it can be readily seen that the total work done by them for any small displacement is zero: from this the statical principle of virtual velocities can be immediately deduced.

Examples.

1. Prove that the whole work done in raising a system of heavy bodies, each through a different height, is the same as that of raising their entire weight through a height equal to that through which their centre of inertia is raised.

2. Find the work performed in moving a ton along 100 yards on a uniformly rough horizontal road, the coefficient of friction being $\frac{1}{10}$.

Ans. 67,200 foot-pounds.

3. Show that the same work is expended in drawing a body up an inclined plane, subject to friction, as would be expended upon drawing it first along the base of the plane (supposing the coefficient of friction the same), and then raising it up the height of the plane.

4. What time will 10 men take to pump the hold of a ship which contains 30,000 cubic feet of water; the centre of inertia of the water being 14 feet below the point of discharge, and each man being supposed to perform 1500 foot-pounds per minute; assuming the weight of a cubic foot of water to be $62\frac{1}{2}$ lbs.?

Ans. 29 hrs. 10 mins.

124. **Work done by a Variable Force.**—If the force be not constant, we may suppose the path described by its point of application divided into portions so small that for each the force may be considered constant. Hence, for the displacement ds of its point of application, Pds is the corresponding *element of work*, and the total work in moving through any space s is represented by the definite integral $\int_0^s Pds$.

If the direction of P makes an angle θ with ds, the corresponding element of work is $P\cos\theta ds$, and the total work is represented by

$$\int_0^s P\cos\theta ds.$$

Again, let x, y, z, be at any instant the coordinates of the point of application of the force P, referred to a system of rectangular axes; and let X, Y, Z, be the components of P parallel to the coordinate axes respectively; then we have

$$P\cos\theta\, ds = Xdx + Ydy + Zdz.$$

Hence the total work done by P in moving its point of application from one point to another is represented by

$$\int(Xdx + Ydy + Zdz)$$

taken between the two points.

If the expression $Xdx + Ydy + Zdz$ be an exact differential, *i.e.* if

$$Xdx + Ydy + Zdz = du,$$

where u is a function of x, y, z, then the integral

$$\int(Xdx + Ydy + Zdz),$$

taken between any two points, is a function of the coordinates of those points; and the work done is accordingly a function

of the extreme coordinates solely. When this is so, the mutual forces between the parts of a system always perform or always consume the same amount of work during any motion whatever by which it can pass from the one particular configuration to the other; hence such a system is called a *conservative system* of forces. In general, for any system of forces acting at different points, the total work W done for any finite displacements is represented by

$$W = \Sigma\int Pdp = \Sigma\int(Xdx + Ydy + Zdz), \qquad (1)$$

where the summation extends to all the forces of the system.

125. **Forces directed to Fixed Centres. Potential.**—If the force F be directed to a fixed centre, and if r be the distance of its point of application from the centre, then the corresponding element of work is represented by Fdr; and the total work, when the point is moved from a distance r' to a distance r'', is represented by $\int_{r'}^{r''} Fdr$.

If F be a function of r represented by $\mu\phi'(r)$, then the value of this integral will be

$$\mu\{\phi(r'') - \phi(r')\}.$$

In the law of attraction which holds in nature we have $F = -\frac{\mu}{r^2}$; and the expression $\mu\left(\frac{1}{r''} - \frac{1}{r'}\right)$ represents the corresponding work in moving a unit of mass from the distance r' to the distance r''. Hence the work done in the motion of a unit mass from an infinite distance to the distance r is represented by $\frac{\mu}{r}$.

The function $\Sigma\frac{m}{r}$ in the case of the ordinary law of gravitation is called the *potential* of the system of attracting masses. This potential function is usually represented by V; and if dm be the element of attracting mass, and r its distance from a point P, then V, the potential at P, is denoted by

$$V = \Sigma\frac{dm}{r} \qquad (2)$$

extended through all points in the attracting system.

Again, if a number of forces F, F', F'', &c., be directed to fixed centres, and if r, r', r'', &c., be the corresponding distances, then the total work is represented by

$$\int Fdr + \int F'dr' + \int F''dr'' + \&c.,$$

taken between the limiting positions.

If the forces be each a known function of the distance from the corresponding centre of force, the result can, in general, be immediately integrated, and the work is a function of the initial and final positions of the points of application solely. Consequently such a system of forces is always a conservative system.

EXAMPLE.

If m, m' be the masses of two particles attracting each other with a force $\mu \dfrac{mm'}{r^2}$, where r is their distance apart, show that the work done when they have moved from an infinite distance apart to the distance r is $\mu \dfrac{mm'}{r}$.

126. **Potential of an Attracting Spherical Mass.**— If each element of the surface of a sphere be divided by its distance from an external point, and the sum taken over the entire surface, this sum is readily shown by elementary integration to be equal to

$$\frac{S}{R},$$

where S is the whole surface of the sphere, and R the distance from its centre to the external point.

Hence, if a mass m be uniformly spread over the surface of the sphere S, we have

$$\Sigma \frac{dm}{r} = \frac{m}{R}. \qquad (3)$$

From this it follows at once that in a solid sphere of mass M, for which the density is constant through each concentric *couche*, we have

$$V = \Sigma \frac{dm}{r} = \frac{M}{R}. \qquad (4)$$

That is, the potential is the same as if the whole mass were concentrated at the centre of the sphere.

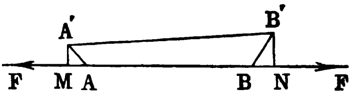
Consequently the work done by an attracting sphere M,

in moving a unit of mass from the distance R' to the distance R, measured from the centre, is

$$W = \mu M \left\{ \frac{1}{R} - \frac{1}{R'} \right\}. \tag{5}$$

It may be remarked that it can be readily seen from (4) that a homogeneous sphere attracts an external mass as if the whole mass of the sphere were concentrated at its centre.

127. **Work done by a Stress.**—If two equal and opposite forces, each represented by F, act respectively at the points A and B, along the line connecting these points, to find the element of work for a small displacement. Suppose A' and B' to be the new positions for an indefinitely small displacement, and let fall the perpendiculars $A'M$ and $B'N$ on the line AB; then the elements of work are represented by $F.AM$ and $F.BN$. Hence their sum is $F(AM + BN) = F(A'B' - AB)$, or $F\Delta s$, where Δs denotes the indefinitely small change in the distance between the points of application of the forces.



Hence, if the points A and B be rigidly connected, as the distance AB is invariable, the total work done by the forces for any displacement is zero.

Also the point of application of a force may be transferred from any one point to any other on its line of action without altering the work done, provided the distance between the two points is invariable.

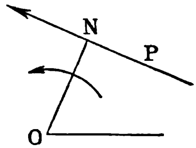
The pair of equal and opposite forces that two bodies exert on one another in accordance with the general principle of action and reaction is called in modern treatises a *stress*. When the forces act away from each other, as in the figure, the stress is called a *tension;* when they act towards each other it is called a *pressure*.

Hence the work done by a stress is positive or negative according as the change of distance between the points of application is in the direction of the mutual action of the forces or in the opposite direction.

Also in the case of a *rigid body* it follows that the *total work done by the internal forces of stress is always zero.*

128. **Body with a Fixed Axis.**—To find the work done by a force acting on a rigid body which is capable of turning round a fixed axis.

Suppose the force R resolved into two components—one parallel, the other perpendicular to the fixed axis. The former does no work, since it is perpendicular to the direction of motion of every point in the body.



Let the latter component be represented by P, and suppose it to act in the plane of the paper; the fixed axis being perpendicular to that plane, and meeting it in the point O. Let N be the foot of the perpendicular drawn from O to the line of action of P; then by the last Article we may take N as the point of application of P.

Suppose now the body to receive a small angular displacement $\Delta\theta$ round the fixed axis in the direction of the arrow; then, if $ON = p$, the displacement of N will be $p\Delta\theta$, and the corresponding element of work is $Pp\Delta\theta$, or $\Delta\theta$ multiplied by the moment of the force R with respect to the fixed axis.

Again, if we suppose a pair of equal, parallel, and opposite forces to act on the rigid body; then, provided the plane of the pair is perpendicular to the fixed axis, the work done by the pair is evidently, from what precedes, represented by the moment of the pair multiplied by the small angle of rotation. And if the pair continue to act on the body, the work done by it during any rotation is represented by the product of the moment of the pair by the angle, in circular measure, through which the body has rotated.

Example.

A pivot or screw turns round a central axis and presses against a rough plane; find an expression for the work expended on the friction which acts on the circular end of the pivot in one revolution round its axis.

Let Q denote the entire normal pressure between the pivot and the plane, μ the coefficient of friction, supposed constant, a the radius of the end of the pivot. This end may be regarded as consisting of an indefinitely great number of concentric circular rings. If r be the radius of one of the rings, dr its breadth, then the area of the ring is $2\pi r\,dr$, and the corresponding friction, taken over the entire ring, is represented by $\frac{2\mu Q}{a^2} r\,dr$. Hence the corresponding

work for one revolution is $\frac{4\mu\pi Q}{a^2}r^2\,dr$. Integrating, we get $\frac{4}{3}\pi\mu Qa$ for the entire work expended. In this investigation the normal pressure Q has been supposed to be uniformly distributed over the end of the pivot.

129. **Energy.**—Energy is the capacity of doing work. For instance, a spring when bent by pressure contains a certain amount of energy stored up in it; thus the mainspring of a watch, by the energy which it possesses, maintains the motions of the works until that energy has been expended. Again, a quantity of air, when compressed into a smaller volume, possesses energy, and can perform work when occasion requires; for example, in projecting a bullet from an air-gun. Also a raised weight is capable of doing work, and is therefore said to possess energy. For instance, the motion of a clock is maintained by the energy of its descending weights. The energy of a weight W raised to a height h above the ground is measured by Wh, that is, by the work it is capable of performing by its descent to the ground. In general, when the configuration of a system is altered, it has a tendency to return to its former state, and in effecting this return is capable of doing a certain amount of work. This capacity of doing work, arising from change of configuration or of relative position in a system, is called *potential energy;* the work employed in producing this change being in a sense accumulated. For example, if two bodies which attract one another are separated, they have a tendency to rush together, and in so doing are capable of overcoming a certain amount of resistance.

Again, a body in motion possesses a certain amount of energy which is measured by the work it is capable of performing before being brought to rest. This latter is called the *Kinetic energy* of the body. We proceed to consider how its amount is measured.

130. **Measure of Kinetic Energy.**—The measure of the kinetic energy of the mass m moving, without rotation, with the velocity v, is easily found. For, suppose the mass acted on by a uniform resistance R in the direction of its motion, and let $R = mf$; then, if v be the initial velocity and s the space described before coming to rest, we have, by Art. 37, $v^2 = 2fs$; hence $\frac{1}{2}mv^2 = Rs$.

Accordingly, the work which a mass m moving with the velocity v is capable of performing before being brought to rest is $\frac{1}{2}mv^2$. Hence its *kinetic energy is equal to half its vis viva*, and is represented by $\frac{1}{2}mv^2$.

Examples.

1. A train of 60 tons, moving at the rate of 15 miles an hour on a horizontal railway, runs, when the steam is shut off and the breaks applied, through a quarter of a mile before stopping. Find in lbs. the mean resistance, and its time of action. *Ans.* 770 lbs.; 2 minutes.

2. The breadth of a river at a certain place is 100 yards, its mean depth is 8 feet, and its mean velocity 3 miles an hour. Calculate its horse-power, assuming a cubic foot of water to weigh $62\frac{1}{2}$ lbs.

Here the quantity of water which passes per minute is 633,600 cubic feet; and the required answer is easily seen to be 363 horse-power.

3. A shot of 1000 lbs., moving at 1600 feet per second, strikes a fixed target. How far will the shot penetrate, the target exerting on it an average pressure equal to the weight of 12,000 tons? *Ans.* $1\frac{1}{2}$ ft., approximately.

4. Determine in ergs the kinetic energy of a mass of one hundred pounds moving with a velocity of one foot per minute. *Ans.* 5853.

5. A heavy particle resting on a rough inclined plane, and attached by a string to a fixed point on the plane, is projected from the lowest point of the circle in which it moves in the direction of the tangent. (*a*) Find the velocity necessary to carry the string to a horizontal position; (*b*) If the particle descending from this position reach the lowest point and remain there, determine the coefficient of friction.

6. A ball moving with a velocity of 1000 feet per second has its velocity reduced by 100 feet in passing through a plank. Through how many such planks would it pass before being stopped; assuming the same amount of work to be performed in overcoming the resistance of each plank? *Ans.* $5\frac{5}{19}$.

131. **Energy due to a Variable Force.**—If a variable force F act at the centre of inertia of a mass m, in the direction of its motion, we have, by Art. 68,

$$F = m\frac{dv}{dt} = mv\frac{dv}{ds},$$

or

$$F ds = mv\,dv;$$

accordingly, if V_0 and V_1 be the initial and final velocities of m, we have

$$\tfrac{1}{2}m(V_1^2 - V_0^2) = \int_0^s F ds. \qquad (6)$$

From this we infer that if a variable force F act on a mass m, in the direction of its motion, the work done by it is measured by half the corresponding change in the *vis viva* of the moving body, or by the change in its kinetic energy.

In general, let X, Y, Z, as before, denote the components, parallel to the axes of x, y, z, of the force acting on the mass m; then, by Art. 68, we have

$$X = m\frac{d^2x}{dt^2}, \quad Y = m\frac{d^2y}{dt^2}, \quad Z = m\frac{d^2z}{dt^2}.$$

Multiply the first by dx, the second by dy, and the third by dz, and add; then

$$\begin{aligned} Xdx + Ydy + Zdz &= m\left(\frac{d^2x}{dt^2}dx + \frac{d^2y}{dt^2}dy + \frac{d^2z}{dt^2}dz\right) \\ &= \tfrac{1}{2}md\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} \\ &= \tfrac{1}{2}md(v^2). \end{aligned}$$

Hence, if V_0 and V_1 be the initial and final velocities,

$$\tfrac{1}{2}m(V_1^2 - V_0^2) = \int(Xdx + Ydy + Zdz), \qquad (7)$$

the integral being taken from the initial to the final position of the centre of inertia of m. Hence we infer that in this case also the work done by the forces during any motion is measured by half the change in the *kinetic energy* of the moving mass.

If after the lapse of any time the velocity of m become equal to its original value, the work done in that interval by the forces which accelerate the motion is equal to that done by the forces which retard it.

In the case of a central force, represented, as in Art. 125, by $\mu\phi'(r)$, we readily obtain the equation

$$\tfrac{1}{2}m(v^2 - v'^2) = \mu\{\phi(r) - \phi(r')\}, \qquad (8)$$

where v' denotes the velocity at the distance r' from the centre of force. For the law of nature, this becomes

$$\tfrac{1}{2}m(v^2 - v'^2) = \mu(V - V'), \qquad (9)$$

where V and V' are the *potentials* of the attraction at the distances r and r', respectively.

Again, in any conservative system of forces the change in the kinetic energy of the motion under the action of the forces, from any one point to any other, is a function of the coordinates of the points, and is independent of the path described.

132. **Equation of Energy.**—In general, if we suppose any free rigid body acted on by external forces, then the total work done by the external forces during any time is equal to the corresponding change of the kinetic energy of the body.

For each particle of the body moves in the same manner as if it were free and acted on by forces equal to those which result from its connexion with the other particles. Hence, by what precedes, the change in the kinetic energy of the particle is equal to the work done on it by the external forces, together with the work due to the stresses which arise from the action of the other particles of the body on it. Accordingly, the total change in the kinetic energy of the rigid body in any time is measured by the work done by the external forces in that time; since, by Art. 127, the internal stresses in this case do no work, and equation (7) may be written in the generalized form

$$\tfrac{1}{2}\Sigma m\,(v^2 - v_0^2) = \Sigma\int(Xdx + Ydy + Zdz), \qquad (10)$$

taken between proper limits, in which the sign of summation, Σ, is extended to each element in the body.

Examples.

1. A locomotive of 10 tons, setting out from rest, acquires a velocity of 20 miles an hour on a horizontal railway, after running through a mile under the action of a constant pressure. Calculate in pounds the difference between the moving and retarding forces, approximately. *Ans.* 57.

2. A 50 lb. ball, after traversing the barrel of a gun of 5 feet length, leaves it with a velocity of 500 feet per second. Find approximately the difference between the mean explosive force of the powder and mean resistance which acts on it. *Ans.* 39062·5 lbs.

3. A uniform block of given dimensions stands, with one face perpendicular to the direction of motion, on a railway truck, which is suddenly stopped. If the block be prevented sliding upon the truck, determine the speed of the train so that the block shall be just overturned.

Here the kinetic energy of the block is expended in raising its centre of gravity until it is vertically over the edge round which the block turns. Accordingly, if a be the height of the block, and b the length of its edge which lies in the direction of motion, the required velocity v is given by the equation

$$v^2 = g\,(\sqrt{a^2 + b^2} - a).$$

4. A catapult is formed by fixing the ends of an elastic string (natural length $2l$) at points A and A', at a short distance apart on a horizontal plane. A bullet placed at the middle point of the string is drawn back at right angles to AA' (stretched length $= 2l'$), and let go when the string is on the point of breaking. Prove that the velocity V of the bullet when it leaves the string is independent of the distance AA', and is to the velocity V' it would have acquired in falling through the vertical space $l' - l$ in the sub-duplicate ratio of the greatest strain W' the string can bear to the weight W of the bullet. Mr. Whitworth, *Educ. Times.*

Let v be the velocity of the bullet, and $2r$ the length of the string at any instant during the motion; then adopting Hooke's law, that the tension of the string varies directly as its extension, the equation of work becomes

$$\frac{W}{g}V^2 = 4\int_l^{l'} \frac{W'}{l'-l}(r-l)\,dr = 2W'(l'-l),$$

or

$$WV^2 = 2g(l'-l)W' = W'V'^2.$$

5. The following extension of the last question is given by Mr. Townsend. If in place of a single cord there be n uniform cords, of the common unextended length $2l$, attached to as many pairs of diametrically opposite points on the circumference of a fixed circle, and all drawing the bullet along the axis of the cone of which the circle is the base, and the bullet at the vertex; then we shall have

$$WV^2 = 2ngW'(l'-l) = W'V'^2,$$

where V' is the velocity due to the height $n(l'-l)$.

133. **Energy of Rotation.**—To find the kinetic energy of a rigid body revolving round a fixed axis with an angular velocity ω.

Let p be the distance from the fixed axis of any element dm of the body; then $p\omega$ will be the velocity of dm, and accordingly the entire *vis viva* of the body

$$\Sigma v^2 dm = \omega^2 \Sigma p^2 dm = \omega^2 I, \qquad (11)$$

where I represents the moment of inertia of the rigid body relative to the fixed axis (*Int. Calc.*, Art. 196). Thus the kinetic energy required is $\frac{1}{2}I\omega^2$.

Examples.

1. The rim of a fly-wheel, sp. gr. 7·25, performing 6 revolutions per minute, is 6 inches thick, and its inner and outer radii are 4 and 5 feet respectively; calculate its kinetie energy in foot-pounds.

Here $\omega = \frac{\pi}{5}$, and M, the mass of the fly-wheel $= 7{\cdot}25 \times \frac{9}{2} \times 62\frac{1}{2} . \pi$. lbs. Also (*Int. Calc.*, Art. 201), $I = M(\frac{41}{2})$; hence the required answer is 805 foot-pounds, approximately.

2. A rod of uniform density can turn freely round one end; it is let fall from a horizontal position; find its angular velocity when it is passing through the vertical position. *Ans.* $\sqrt{\frac{3g}{a}}$, where a is the length of the rod.

3. Two masses M and M' are connected as in Atwood's machine (Art. 78); find the acceleration when the mass μ of the revolving pulley is taken into account. If v be the common velocity of M and M' at any instant, and μk^2 the moment of inertia of the pulley; then the entire *vis viva* of the system is represented by $(M + M') v^2 + \mu k^2 \omega^2$.

Hence, if z be the distance fallen through from rest, we have

$$(M + M') v^2 + \mu k^2 \omega^2 = 2g (M - M') z.$$

Also $v = a\omega$;

$$\therefore v^2 \{(M + M') a^2 + \mu k^2\} = 2ga^2 (M - M') z.$$

Again, the acceleration $$f = \frac{dv}{dt} = v\frac{dv}{dz};$$

therefore $$f = \frac{ga^2 (M - M')}{(M + M') a^2 + \mu k^2}.$$

If the pulley be supposed a homogeneous cylinder, $k^2 = \frac{a^2}{2}$, and f becomes

$$\frac{g (M - M')}{M + M' + \frac{1}{2}\mu}.$$

4. Find in the same case the tensions of the strings.

Ans. $$Mg\frac{2M' a^2 + \mu k^2}{(M + M') a^2 + \mu k^2}, \quad M'g\frac{2Ma^2 + \mu k^2}{(M + M') a^2 + \mu k^2}.$$

For a homogeneous pulley these become

$$Mg\frac{4M' + \mu}{2(M + M') + \mu}, \quad \text{and} \quad M'g\frac{4M + \mu}{2(M + M') + \mu}.$$

5. A homogeneous cylinder, of weight W, is rotating round its axis, supposed horizontal, with an angular velocity ω; find to what height it is capable of raising a given weight P, before coming to rest.

Ans. $\frac{r^2 \omega^2}{4g}\frac{W}{P}$, where r is the radius of the cylinder.

134. **Vis Viva of any System.**—If $\bar{x}, \bar{y}, \bar{z}$ be the coordinates of the centre of gravity of any moving system of masses at any instant, x, y, z the coordinates of the element dm at the same instant; also, if ξ, η, ζ be the coordinates of dm relative to a system of parallel axes drawn through the centre of gravity; then, as in Art. 14, we have, adopting Newton's notation,

$$\dot{x} = \dot{\bar{x}} + \dot{\xi}, \quad \dot{y} = \dot{\bar{y}} + \dot{\eta}, \quad \dot{z} = \dot{\bar{z}} + \dot{\zeta};$$

consequently,

$$\Sigma v^2 dm = \Sigma dm \{(\dot{\bar{x}} + \dot{\xi})^2 + (\dot{\bar{y}} + \dot{\eta})^2 + (\dot{\bar{z}} + \dot{\zeta})^2\}.$$

Again, if V be the velocity of the centre of gravity, and v' the velocity of dm relative to the centre of gravity, we have

$$V^2 = (\dot{\bar{x}})^2 + (\dot{\bar{y}})^2 + (\dot{\bar{z}})^2, \quad v'^2 = \dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2;$$

also

$$\Sigma\dot{\xi}\, dm = 0, \quad \Sigma\dot{\eta}\, dm = 0, \quad \Sigma\dot{\zeta}\, dm = 0.$$

Hence we get

$$\Sigma v^2\, dm = V^2 \Sigma\, dm + \Sigma v'^2\, dm. \tag{12}$$

Accordingly, the *vis viva* of the system at any instant consists of two parts, one of which is the *vis viva* of the entire mass supposed concentrated at the centre of gravity; the other is the *vis viva* of the system relative to the centre of gravity regarded as a fixed point. This result obviously holds good also for the *kinetic energy* of the motion.

Examples.

1. A homogeneous cylinder rolls, without slipping, down a rough inclined plane, under the action of gravity; investigate the motion.

Since the motion is one of pure rolling, the line of contact of the cylinder and plane at any instant may be regarded as fixed; accordingly the friction acting along the plane does no work. Also, by Art. 133, the kinetic energy at any instant is represented by $\frac{1}{2}\omega^2 I$, where I is the moment of inertia of the cylinder with respect to the edge in contact with the plane. But $I = M(a^2 + k^2)$, where a is the radius of the cylinder, and Mk^2 its moment of inertia relative to the axis through its centre. Hence the equation of work gives

$$M\omega^2 (a^2 + k^2) = 2gMs \sin i,$$

where s is the space down the plane described from rest. Consequently,

$$\omega^2 = \frac{2gs \sin i}{a^2 + k^2}. \quad \text{Also, } \frac{ds}{dt} = v = a\omega;$$

hence

$$\left(\frac{ds}{dt}\right)^2 = \frac{2ga^2 s \sin i}{a^2 + k^2};$$

therefore, by differentiation,

$$\frac{d^2 s}{d^2 t} = \frac{a^2 g \sin i}{a^2 + k^2}.$$

This shows that the acceleration down the plane is constant. Hence the velocity acquired, and the space described in any time, can at once be determined. If the cylinder be homogeneous, we have $k^2 = \frac{1}{2}a^2$ (*Int. Calc.*, Art. 201), and the acceleration f in this case is $\frac{2}{3}g \sin i$. This shows that the velocity of the centre of gravity of the cylinder is $\frac{2}{3}$ that acquired by a particle, in the same time, in sliding down a smooth inclined plane of the same inclination. If the cylinder be hollow, $k = a$, and accordingly $f = \frac{1}{2}g \sin i$.

2. A mass M draws up another, M', on the wheel and axle; find the motion.

Let a be the radius of the wheel, a' that of the axle; then, as in Ex. 3, Art. 133, it is easily seen that we get

$$\left(\frac{d\theta}{dt}\right)^2 (Ma^2 + M'a'^2 + \mu k^2) = 2g(Ma - M'a')\theta + \text{const.}$$

Hence, by differentiation,

$$\frac{d^2\theta}{dt^2} = \frac{g(Ma - M'a')}{Ma^2 + M'a'^2 + \mu k^2}.$$

Accordingly, if $\theta = 0$, and $\frac{d\theta}{dt} = 0$, when $t = 0$, we get for the angle turned through in the time t,

$$\theta = \tfrac{1}{2}gt^2 \frac{Ma - M'a'}{Ma^2 + M'a'^2 + \mu k^2}.$$

3. Find the tensions of the strings in the same case.

Ans. $\quad Mg\dfrac{M'a'(a + a') + \mu k^2}{Ma^2 + M'a'^2 + \mu k^2}, \quad M'g\dfrac{Ma(a + a') + \mu k^2}{Ma^2 + M'a'^2 + \mu k^2}.$

4. Find the velocity acquired by the centre of a hoop in rolling down an inclined plane of height h. *Ans.* $\sqrt{gh}$.

135. Work done by an Impulse.—If a mass M moving with a velocity V receives an impulse in the direction of its motion, and if V' be its velocity after the impulse, then the change in its kinetic energy is

$$\tfrac{1}{2}M(V'^2 - V^2) = M(V' - V) \cdot \tfrac{1}{2}(V' + V).$$

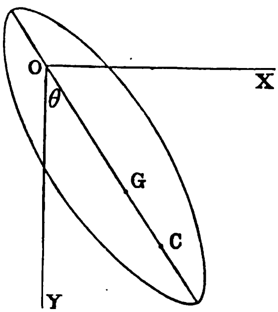
But $M(V' - V)$ measures the impulse. Hence the work done by the impulse is measured by the product of the momentum, which measures the impulse, by half the sum of the velocities before and after the impulse.

For example, a bullet m in passing through a plank experiences a definite amount of resistance, measured by the thickness and by the resisting force; but this equals half the loss of *vis viva* of the bullet, or

$$\tfrac{1}{2}m(v^2 - v'^2) = m(v - v') \cdot \tfrac{1}{2}(v + v'),$$

where v and v' are the velocities with which it meets and leaves the plank. Hence the momentum $m(v - v')$ communicated to the plank varies inversely as $v + v'$: consequently the greater the velocity of impact the less the momentum imparted. This explains how a bullet with a high velocity can pass through a door without moving it on its hinges.

136. **Compound Pendulum.**—A solid body oscillating under the action of gravity, around a fixed horizontal axis, is called a *compound pendulum*. The motion of such a body is readily reduced to that of the corresponding simple pendulum, as follows:



Let the plane of the paper represent that in which the motion of G, the centre of inertia of the body, takes place, and let O be the point in which the fixed axis intersects that plane. Draw OY vertically downwards, and let $GO = a$, M = mass of the body. Also let $\angle GOY = \theta$.

Suppose the pendulum to start from rest, when $\theta = \alpha$; then, in the time t, the point G will have descended through the vertical height $a(\cos\theta - \cos\alpha)$. Also the *vis viva* of the body at the same instant (Art. 133) is represented by

$$\left(\frac{d\theta}{dt}\right)^2 \Sigma r^2 dm = I\left(\frac{d\theta}{dt}\right)^2.$$

Hence, by the principle of work, Art. 132, we have

$$I\left(\frac{d\theta}{dt}\right)^2 = 2Mga(\cos\theta - \cos\alpha).$$

If the moment of inertia I be represented by MK^2, the latter equation becomes

$$K^2\left(\frac{d\theta}{dt}\right)^2 = 2ga(\cos\theta - \cos\alpha),$$

where K is the radius of gyration of the body (*Int. Calc.*, Art. 197), relative to the axis of suspension.

Hence, by differentiation,

$$\frac{d^2\theta}{dt^2} + \frac{ga}{K^2}\sin\theta = 0. \tag{13}$$

Comparing this with the corresponding equation for the motion of a simple pendulum (Art. 101), we see that the motion is the same as that of a simple pendulum of length $l = \frac{K^2}{a}$.

Again, if Mk^2 be the moment of inertia relative to an axis through the centre of inertia parallel to the axis of suspension, we have (*Int. Calc.*, Art. 196),

$$K^2 = a^2 + k^2;$$

hence
$$l = \frac{K^2}{a} = a + \frac{k^2}{a}. \tag{14}$$

The point O is called the *centre of suspension.* If OG be produced until $OC = l$, since the body moves as if its entire mass were concentrated at the point C, that point is called the *centre of oscillation.* Again, if through C a right line be drawn parallel to the axis of suspension, all the points of this line move like the point C, i. e. as if they were freely suspended from the axis of rotation. This line is called the *axis of oscillation.*

Again, since $OG \cdot GC = k^2$, the axes of suspension and oscillation are interchangeable, *i. e.* the time T of an oscillation is the same for both, viz., $T = \pi\sqrt{\frac{a^2 + k^2}{ag}}$.

By varying the axis of suspension, the time of a small oscillation will also, in general, vary.

For parallel axes, T obviously is a minimum when $a = k$, and the corresponding time of a small oscillation $= \pi\sqrt{\frac{2k}{g}}$.

In order that this should be the smallest possible, the axis of suspension must be parallel to that axis round which the moment of inertia is least (*Int. Calc.*, Art. 217).

If the axis of suspension of a compound pendulum be inclined at an angle α to the vertical, it is readily seen that the preceding investigation holds good, provided $g \sin \alpha$ be substituted for g throughout.

Again, as in Art. 101, the time of any motion of a compound pendulum is represented by an elliptic integral.

Also, if a solid body make a complete revolution round a horizontal axis, the time of revolving through any angle can be reduced to that for the corresponding oscillatory motion of a particle.

Examples.

1. A uniform circular plate, of radius a, makes small oscillations about a horizontal tangent; find the length of the equivalent simple pendulum. *Ans.* $\frac{5}{4}a$.

2. Find the position of the axis with respect to which a uniform circular plate will oscillate in the shortest time.

Ans. The axis is at a distance of half the radius from the centre. Length of the equivalent pendulum $= a$.

3. Find the centre of oscillation of a homogeneous sphere, of radius a, oscillating round a horizontal tangent to its surface.

Ans. At a point $\frac{2}{5}a$ below the centre.

4. Find the ratio of the times of oscillation of a homogeneous solid sphere, and of a spherical shell of equal diameter, each being taken with reference to a horizontal tangent. *Ans.* $\sqrt{21} : 5$.

5. A sphere of radius a is suspended by a fine wire from a fixed point, at a distance l from its centre; prove that the time of a small oscillation is represented by $\pi\sqrt{\frac{5l^2 + 2a^2}{5lg}}\,(1 + \frac{1}{4}\sin^2 \frac{1}{2}\alpha)$, where α represents the amplitude of the vibration.

6. If the semiaxes of a uniform elliptic disc be 2 feet and 1 foot, and it be suspended from an axis perpendicular to its plane through one of its foci, find the time of a complete oscillation under gravity. *Ans.* $\frac{\pi}{2\sqrt[4]{3}}\sqrt{\frac{17}{g}}$.

137. Determination of the Force of Gravity.—We have already seen (Art. 103) that the value of g at any place can be determined from the length of the seconds pendulum at the place. To apply this it is necessary to know the numerical value of $\frac{a^2 + k^2}{a}$.

Two methods have been devised for this purpose—one employed by Borda, Arago, Biot, and others; the other first

used by Bohnenberger, and afterwards brought to great perfection by Captain Kater.

In the first method the compound pendulum, supposed made of a material of uniform density, has such a shape that its radius of gyration can be calculated mathematically, as also the distance of its centre of inertia from the fixed axis.

The second method depends on the reciprocity of the centres of suspension and oscillation.

Kater's compound pendulum consisted of a heavy bar having two apertures at opposite sides of the centre of inertia, through which knife edges passed, on either of which the body could be supported. On the bar was placed a ring capable of being moved up or down by means of a screw. Kater moved the ring until the times of oscillation round the two axes were equal; in which case, by the preceding, the distance between the axes is equal to the length of the equivalent simple pendulum. The distance, l, between the axes having been accurately measured, the value of g was calculated from the formula $g = \frac{\pi^2 l}{T^2}$, where T denotes the time of an oscillation.

Kater published an account of his observations in the *Philosophical Transactions*, 1818, 1819. For a more detailed account of this method the reader is referred to Routh's *Rigid Dynamics*, Arts. 100–108.

138. **Motion of a Rigid Body round a Fixed Axis.** —In general, let a force P, in a direction which is at right angles to the fixed axis, act on a body; then for a small angular motion $d\theta$ the work done by P is, by Art. 128, represented by $Pp\,d\theta$. Again, as this work is equal to the corresponding change in the *kinetic* energy of the body, we have

$$Pp\,d\theta = \tfrac{1}{2}Mk^2 d\left(\frac{d\theta}{dt}\right)^2 = Mk^2\frac{d^2\theta}{dt^2}\,d\theta.$$

Hence we get

$$\frac{d^2\theta}{dt^2} = \frac{Pp}{Mk^2} = \frac{\text{Moment of impressed force}}{\text{Moment of inertia}}. \qquad (15)$$

Examples.

1. A uniform circular plate of 1 foot radius and 1 cwt. revolves round its axis 5 times per second; calculate its kinetic energy in foot pounds.

Ans. 863, approximately.

2. A bent lever ACB rests in equilibrium when AC is inclined at the angle α to the horizontal line; show that when this arm is raised to the horizontal position it will fall through the angle 2α, C being supposed fixed.

3. A homogeneous cylinder, of mass M, and radius a, turns round a horizontal axis; a fine thread is wrapped round it, and has a mass M' attached to its extremity. Find the angular velocity of the cylinder when M' has descended through the height h.

Ans. $\omega^2 = \dfrac{4M'gh}{a^2(M + 2M')}$.

4. A right cone oscillates round a horizontal axis, passing through its vertex and perpendicular to the axis of the cone; find the length of the equivalent simple pendulum.

Ans. $\dfrac{4h^2 + b^2}{5h}$, where h is the height of the cone, and b the radius of its base.

5. If in the last example the cone be let fall from the position in which its axis is horizontal, find its angular velocity when in the lowest position.

Ans. $\omega^2 = \dfrac{10hg}{4h^2 + b^2}$.

6. In the same case find the pressure on the fixed axis, at the lowest position of the body, arising from centrifugal force (Art. 98).

Ans. $\dfrac{15}{2} W \dfrac{h^2}{4h^2 + b^2}$, where W represents the weight of the cone.

7. A thin beam, whose mass is M and length $2a$, moves freely about one extremity attached to a fixed point whose distance from a smooth plane is b, ($b < 2a$): the other extremity rests on the plane, the inclination of which is α. If the beam be slightly displaced from its position of equilibrium determine the time of its small oscillations.

Indian Civil Service Exam., 1860.

In this case the beam may be regarded as turning round the perpendicular on the plane.

8. A bullet weighing 50 grammes is fired into the centre of a target with a velocity of 500 metres a second. The target is supposed to weigh a kilogramme, and to be free to move. Find, in kilogrammetres, the loss of energy in the impact.

Lond. Univ., 1880. *Ans.* 635·6.

9. When the weight P of the pulley is taken into account, show that equation (9), Art. 76, becomes

$$\mu = \frac{W}{W'} - \frac{f}{g}\left(1 + \frac{W + \frac{1}{2}P}{W'}\right),$$

in which the pulley is supposed to be of uniform density and thickness.

10. If the motion of a solid body acted on by attracting forces be a pure rotation, the velocity ω of rotation at any instant will be given by the equation

$$Mk^2(\omega^2 - \omega_0^2) = 2(V - V_0),$$

where V represents the potential of the attracting forces.

11. A hollow cylinder rolls down a perfectly rough inclined plane in 10 minutes; find the time a uniform solid cylinder would take to roll down the same plane. *Ans.* $5\sqrt{3}$ minutes.

12. The particles composing a homogeneous sphere of mass M and radius R were originally at an infinite distance from each other: find the work done by their mutual attraction.

Suppose the sphere in question to have been formed by the condensation of an indefinitely diffused nebula; and imagine the sphere divided into a number of concentric spheres. Let M' be the mass contained in the sphere whose radius is r; then we have

$$M' = M\frac{r^3}{R^3}.$$

Also, if dM' be the mass bounded by the spheres r and $r + dr$, then

$$dM' = 3M\frac{r^2\,dr}{R^3}.$$

Accordingly the work done in condensing dM', in consequence of the attraction of the *interior mass* M', is, by (5) Art. 126,

$$\mu\frac{M'}{r}\,dM' = 3\mu\frac{M^2}{R^6}r^4\,dr.$$

Hence the whole work done in the condensation of M is

$$3\mu\frac{M^2}{R^6}\int_0^R r^4\,dr = \frac{3}{5}\mu\frac{M^2}{R}.$$

CHAPTER VII.

CENTRAL FORCES.

SECTION I.—*Rectilinear Motion.*

139. **Centre of Force.**—We next proceed to consider motion under the action of a force whose direction always passes through a fixed point, and whose intensity is a function of the distance from that point. The fixed point is called the *Centre of Force;* and the force is said to be attractive or repulsive according as it is directed towards or from the centre.

If we assume that two particles of equal mass, placed at the same distance from a centre of attractive force, are equally attracted towards the centre, when they are conceived placed *together*, the whole force acting on them—considered as one mass—will be double that which acts on one of the particles. Similarly, if any number (n) of equal particles be placed together, the whole force will be n times that which acts on a single particle. Hence it follows that in such cases the whole attracting force is proportional to the number of particles, *i. e.* to the *mass* of the attracted body—provided the attracted mass be of such small dimensions that the lines drawn from its several points to the centre of force may be regarded as equal and parallel. Accordingly the force, in this case, is proportional to the attracted mass; consequently the *acceleration* produced by it is independent of the mass attracted, and is a function of the distance from the centre of force only.

140. **Attraction.**—The acceleration due to an attractive force, at any distance, is called the *attraction* of the force, and is, as we have seen, independent of the mass of the attracted particle. Consequently the measure of an attractive force at any distance is the velocity per second which the

central force could generate in one second, in its own direction, if it were conceived to act uniformly during that time. For instance, g, i. e. the velocity acquired in one second by a falling body (Art. 38), measures the attractive force of the Earth, at any place, and is, as already stated, the same for all bodies at that place.

141. **Rectilinear Motion.**—If the particle acted on be originally at rest, or be projected in the line joining its position to the centre of force, its motion will take place in that right line.

Taking this line for the axis of x, and the fixed centre as origin, we have for the equation of motion (Art. 21),

$$\frac{d^2x}{dt^2} = -F, \qquad (1)$$

where F represents the *attraction* at the distance x, which is taken with the negative sign because it tends to *diminish* the velocity.

We shall illustrate equation (1) by applying it to a few elementary cases.

142. **Force Varying as the Distance.**—If the force be proportional to the distance from the fixed centre, we may assume $F = \mu x$; then, for attractive forces, the equation of motion becomes

$$\frac{d^2x}{dt^2} = -\mu x, \text{ or } \frac{d^2x}{dt^2} + \mu x = 0. \qquad (2)$$

This equation has been already considered in Art. 109, and accordingly we have

$$x = C \cos t\sqrt{\mu} + C' \sin t\sqrt{\mu}. \qquad (3)$$

The constants C and C' are determined from the initial circumstances of the motion.

For example, if the particle start from rest, at the distance a from the centre of force; then, when $t = 0$, we have $x = a$ and $\frac{dx}{dt} = 0$: this gies

$$C = a, \text{ and } C' = 0;$$

and consequently $x = a \cos t\sqrt{\mu}$. This determines the position of the particle at any instant, and shows that the motion consists of a simple harmonic vibration.

Again, if $(t' - t)\sqrt{\mu} = 2\pi$, it is evident that the values of x and of $\frac{dx}{dt}$ are the same at the end of the time t' as at the time t: this shows that the motion is oscillatory, and that the time of a complete vibration is $\frac{2\pi}{\sqrt{\mu}}$. (Compare Art. 111.)

For a repulsive force the equation of motion is

$$\frac{d^2x}{dt^2} = \mu x. \qquad (4)$$

Accordingly (Art. 109), we have

$$x = Ce^{t\sqrt{\mu}} + C'e^{-t\sqrt{\mu}}.$$

To determine the constants: snppose, as in the former case, the particle starts from rest, at the distance a; then

$$a = C + C', \text{ and } C - C' = 0.$$

Hence $$x = \tfrac{1}{2}a\left(e^{t\sqrt{\mu}} + e^{-t\sqrt{\mu}}\right). \qquad (5)$$

143. **Inverse Square of Distance.**—In the case of the law of nature, in which the attractive force varies as the inverse square of the distance, we have $F = \frac{\mu}{x^2}$; and the differential equation of motion is

$$\frac{d^2x}{dt^2} + \frac{\mu}{x^2} = 0.$$

Multiplying by $2dx$, and integrating, we get

$$\left(\frac{dx}{dt}\right)^2 - \frac{2\mu}{x} = \text{const.}$$

Hence, if the particle be supposed to start from rest, at the distance a,

$$\left(\frac{dx}{dt}\right)^2 = 2\mu\left(\frac{1}{x} - \frac{1}{a}\right). \qquad (6)$$

This equation determines the velocity at any distance from the centre of force.

Again, extracting the square root, and transforming, we get

$$\sqrt{2\mu}\,dt = \frac{-dx}{\sqrt{\frac{1}{x} - \frac{1}{a}}}. \qquad (7)$$

The negative sign is taken since, in the motion towards the centre of force, x diminishes as t increases.

To integrate this equation, assume $x = a\cos^2\theta$; then

$$\sqrt{\frac{1}{x} - \frac{1}{a}} = \frac{\tan\theta}{\sqrt{a}}, \text{ and } dx = -2a\sin\theta\cos\theta\,d\theta;$$

consequently $\qquad \sqrt{2\mu}\,dt = 2a^{\frac{3}{2}}\cos^2\theta\,d\theta;$

hence $\qquad t = \sqrt{\frac{a^3}{2\mu}}\,(\theta + \tfrac{1}{2}\sin 2\theta) + \text{constant}.$

Again, the constant vanishes, since t and θ vanish when $x = a$;

$$\therefore\ t = \sqrt{\frac{a^3}{2\mu}}\,(\theta + \tfrac{1}{2}\sin 2\theta). \qquad (8)$$

Hence the time of motion from the distance a to the distance x is

$$t = \sqrt{\frac{a}{2\mu}}\left(a\cos^{-1}\sqrt{\frac{x}{a}} + \sqrt{x(a-x)}\right). \qquad (9)$$

Also the time of motion to the centre of force is

$$\frac{\pi}{2}\sqrt{\frac{a^3}{2\mu}}.$$

Again, if the body be supposed to start from an indefinitely great distance we have, making $a = \infty$ in (6),

$$v^2 = \frac{2\mu}{x}. \tag{10}$$

144. Application to the Earth.—We have seen, in Art. 126, that the attraction of a homogeneous sphere is the same as if its mass were concentrated at its centre. Hence, the results of the last Article can be readily applied to the approximate determination of the motion of a body falling from any height above the Earth's surface, all resistance of the atmosphere being neglected.

In this case g measures the Earth's attraction at its surface; hence, if R denote the Earth's radius, we have $\mu = gR^2$, and if this value be substituted for μ, we can readily determine the velocity and time of motion in any particular case.

For instance, the velocity V with which a body falling from the height h would reach the surface of the Earth is given by the equation

$$V^2 = 2g\frac{Rh}{R+h}. \tag{11}$$

Also, by (9), the time of motion in seconds is

$$\sqrt{\frac{R+h}{2g}}\left\{\frac{R+h}{R}\sin^{-1}\sqrt{\frac{h}{R+h}} + \sqrt{\frac{h}{R}}\right\},$$

where R and h are expressed in feet.

If $R = nh$, this becomes

$$\sqrt{\frac{h}{2g}\frac{(1+n)}{n}}\left\{\frac{1+n}{\sqrt{n}}\sin^{-1}\frac{1}{\sqrt{1+n}} + 1\right\}.$$

When n is a large number this becomes, approximately,

$$\sqrt{\frac{2h}{g}}\left(1 + \frac{5}{6n}\right). \tag{12}$$

If the body be supposed to start from an infinite distance, the velocity with which it would reach the Earth is given by the equation

$$v^2 = 2gR. \tag{13}$$

145. **Comparison of Attraction of Different Spherical Bodies.**—Let M, M' denote the masses of two spheres; δ, δ' their mean densities; r, r' their radii; f, f' their attractions at their surfaces, respectively: then we have

$$f : f' = \frac{M}{r^2} : \frac{M'}{r'^2} = \delta r : \delta' r'.$$

For example, if D be the mean density of the Earth, and R its radius, then f, the attraction at the surface of a planet of radius r and mean density δ, is given by the equation

$$f = g \frac{\delta}{D} \frac{r}{R}. \tag{14}$$

If the mean densities be the same for both, we have

$$f = g \frac{r}{R}.$$

If we assume the mean density of the Sun to be one-fourth that of the Earth, and its radius 104 times that of the Earth, then the velocity acquired in one second by a falling body at the Sun's surface is approximately represented by $26g$.

In the case of the mutual attraction of two spheres, it is often convenient to assume the origin at their common centre of gravity, which remains a fixed point during the motion. For instance, if two equal spheres, each of radius r, be placed at a given distance apart, and left to their mutual attraction, this method may be employed to find the time they would take to come together.

Let $2a$ be the initial distance between their centres, and assume the origin O at the middle point of the line joining the centres. If x be the distance of the centre of either sphere from O at any time; then $\frac{\mu}{4x^2}$ represents the corre-

sponding attraction, and the time required is, by (9), represented by the expression

$$\sqrt{\frac{2a}{\mu}}\left(a\cos^{-1}\sqrt{\frac{r}{a}} + \sqrt{r(a-r)}\right), \tag{15}$$

where μ can be determined by the equation

$$\frac{\mu}{r^2} = f = g\frac{\delta}{D}\frac{r}{R}.$$

EXAMPLES.

1. If h be the height due to the velocity V at the Earth's surface, supposing its attraction constant, and H the corresponding height when the variation of gravity is taken into account, prove that

$$\frac{1}{h} - \frac{1}{H} = \frac{1}{R}.$$

2. If a man weigh 10 stone on the Earth's surface, calculate, approximately, his weight if he were transferred to the surface of the Sun.

Ans. 1 ton, 13 cwt.

3. Calculate, approximately, the velocity with which a body falling from an indefinitely great distance would reach the surface of the Earth, neglecting all forces besides the Earth's attraction, and assuming $R = 4000$ miles.

Ans. 7 miles per second.

4. Calculate, in like manner, the velocity with which a body falling from an indefinitely great distance would reach the surface of the Sun.

Ans. 364 miles per second.

5. In a work erroneously attributed to Sir Isaac Newton, it is stated, that if two spheres, each one foot in diameter, and of a like nature to the Earth, were distant by but the fourth part of an inch, they would not, even in spaces void of resistance, come together by the force of their mutual attraction in less than a month's time.

Investigate the truth of this statement. *Sch. Ex.*, 1883.

Equation (15) gives in this case for the time, in seconds,

$$700\sqrt{11}\left\{\frac{49}{96}\sin^{-1}\left(\frac{1}{7}\right) + \frac{1}{8\sqrt{3}}\right\}.$$

This gives about 5 minutes and 38 seconds.

If the question be solved on the assumption that the attraction is constant during the motion, and equal to that when the spheres are touching, the time required is readily found to be, approximately, $= 100\sqrt{11} = 5$ m. 32 secs.

It may be observed that the former result follows from this immediately by application of formula (12).

Hooke's law, the tension T of the string for the extension $x - a$ is represented by

$$T = mg \frac{x - a}{b}. \tag{19}$$

Accordingly, the equation of motion of the particle is

$$\frac{d^2x}{dt^2} + \frac{g}{b}(x - a) = 0. \tag{20}$$

Integrating, we have

$$x = a + C \cos \sqrt{\frac{g}{b}}\, t + C' \sin \sqrt{\frac{g}{b}}\, t.$$

To determine the constants, let a' denote the *initial length* of the string; then

$$a' = a + C, \text{ i.e. } C = a' - a;$$

also, since $\frac{dx}{dt} = 0$, when $t = 0$, we have

$$C' = 0.$$

Consequently, $x = a + (a' - a) \cos \sqrt{\frac{g}{b}}\, t.$ (21)

This gives the position of the particle so long as the string is stretched, *i. e.* so long as x is greater than a.

The velocity at any instant is given by the equation

$$\frac{dx}{dt} = -(a' - a) \sqrt{\frac{g}{b}} \sin \sqrt{\frac{g}{b}}\, t.$$

The length x becomes equal to a, or the string regains its natural length, and the tension ceases to act at the end of the time $\frac{\pi}{2}\sqrt{\frac{b}{g}}$.

Meanwhile the velocity has increased from zero, and attained its maximum value

$$-(a' - a) \sqrt{\frac{g}{b}},$$

at the same instant.

The particle will now continue to move uniformly along the table with this velocity until it arrives at the same distance a on the opposite side of the fixed extremity of the string, when it becomes again acted on by the retarding tension of the string; and the same motion will be repeated.

148. **Weight Suspended by an Elastic String.**— We shall next consider the vertical oscillations of a body, of weight W, attached to the end of an elastic string, which hangs freely from a fixed point. Suppose the body depressed below the position of equilibrium, and then set at liberty, to investigate the subsequent motion.

As before, let b be the extension of the string due to the weight W; c its extension at the commencement of the motion; x its extension at any instant; T the corresponding tension of the string: then, by Hooke's law, we have

$$T = W\frac{x}{b}, \tag{22}$$

and the differential equation of motion is obviously

$$m\frac{d^2x}{dt^2} = W - T,$$

or
$$\frac{d^2x}{dt^2} + \frac{g}{b}(x - b) = 0. \tag{23}$$

The integral of this is

$$x = b + C\cos\sqrt{\frac{g}{b}}\,t + C'\sin\sqrt{\frac{g}{b}}\,t.$$

To determine the constants, we have, when

$$t = 0,\ x = c,\ \text{and}\ \frac{dx}{dt} = 0;$$

therefore,
$$C = c - b,\ \text{and}\ C' = 0.$$

Consequently,
$$x = b + (c - b)\cos\sqrt{\frac{g}{b}}\,t. \tag{24}$$

There are two cases to be considered, according as c is less or greater than $2b$.

(1). Let $c < 2b$. In this case the extension x, and con-

sequently the tension T, can never vanish; and the body will oscillate up and down through the distance $c - b$, on both sides of the position of equilibrium; the time of an oscillation being represented by

$$\pi \sqrt{\frac{b}{g}}.$$

(2). Next, let $c > 2b$. In this case x vanishes, and consequently T also, when

$$b + (c - b) \cos \sqrt{\frac{g}{b}}\, t = 0.$$

The corresponding velocity is easily found to be

$$\sqrt{\frac{gc\,(c - 2b)}{b}}.$$

As the tension of the string vanishes at this instant, the body may be regarded as projected upwards with the foregoing velocity. The height, h, to which it would ascend is given by the equation

$$h = \frac{c}{2b}\,(c - 2b). \qquad (25)$$

The body will afterwards fall to the origin, and the subsequent motion will be as before.

149. **Weight Dropped from a Height.**—Next suppose the weight attached to the string, and dropped from a height h, vertically above the lower extremity of the string when hanging freely and unstretched. The solution is contained in the preceding investigation: for the maximum extension c of the string is given by (25), and is represented by

$$c = b + \sqrt{b\,(b + 2h)}. \qquad (26)$$

In practice it is found that Hooke's law does not hold beyond certain limits which are attained long before the string is broken. It is interesting to consider whether in any particular case the string will be broken or not by the fall, assuming Hooke's law still to hold.

A given string is capable of supporting only a certain weight, called its *breaking weight*. Denote this weight by B;

then e, the corresponding extension of the string, is found, by Hooke's law, from

$$B = \frac{W}{b} e, \tag{27}$$

and the string will break or not according as the maximum extension, given by the preceding analysis, is greater or less than e; that is, according as $b + \sqrt{b(b + 2h)}$ is greater or less than e.

Again, if b and e be both given, the least height of fall, h, in order that the string should break, is got by substituting e for c in (25), and is

$$h = \frac{e(e - 2b)}{2b}. \tag{28}$$

Suppose the weight W to be the n^{th} part of B, i.e. let $e = nb$, and we have $h = e(\frac{1}{2}n - 1)$.

Thus, for instance, a weight $\frac{1}{4}$ of the breaking weight, dropped from the height e, should suffice to break the string.

The preceding analysis applies also to the vertical oscillations of rods supporting heavy weights; and many interesting practical questions are explained thereby—for instance, the danger to the stability of a suspension bridge arising from the steady march of troops over it.—See Poncelet, *Mécanique Industrielle*, Arts. 332–345.

Examples.

1. A heavy particle attached to a fixed point by an elastic string is allowed to fall freely from this point. Show that the elastic force at the lowest point is given by the equation

$$F = 2W \frac{\text{total fall}}{\text{extension of string}},$$

where W is the weight of the particle.

2. A heavy particle attached to a fixed point by an elastic string hangs freely, stretching the string by a quantity e. It is drawn down by an additional distance f; determine the height to which it will rise if $f^2 - e^2 = 4ae$, a being the unstretched length of the string. *Ans.* $2a$.

3. A heavy body is attached to a fixed point by an elastic string, which passes through a fixed ring, the natural length of the string being equal to the distance between the ring and the fixed point.

(*a*) If the body receive an impulse, it will describe an ellipse round the place it would occupy if suspended freely.

(*b*) When does this ellipse become a right line?

4. A particle is attached by a straight elastic string to a centre of repulsive force, the intensity of which varies as the distance; the string is at first at its natural length. Find the greatest distance from the centre of force to which the particle will proceed, and the time the string takes to return to its natural length.

5. Two bodies, W and W', hang at rest, being attached to the lower end of a fine elastic string, whose upper end is fixed: supposing one of them, W', to drop off, find the subsequent motion of the other.

Let a be the natural length of the string; b its extension of length for the weight W; c that for the weight W'; then, at the end of any time t, from the commencement of the motion x, the depth of W below the fixed point is given by the equation $x = a + b + c \cos t \sqrt{\frac{g}{b}}$.

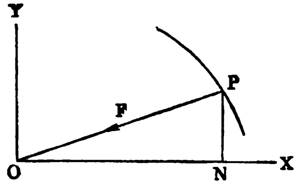
6. Two particles, connected by a fine elastic string, are moving in the direction of the line joining them with equal velocities, their distance being the natural length of the string; if the hinder particle be suddenly stopped, find how far the other will move before it begins to return.

SECTION II.—*Central Orbits.*

150. **Plane of Orbit.**—If we suppose a particle acted on by a force directed to a fixed centre to be projected in any direction, it is easily seen that its subsequent path will lie in the plane passing through the centre of force and the direction of its projection. For, since the force acts towards the fixed centre, it has no tendency to withdraw the particle from that plane at the first instant, nor at any subsequent instant during the motion; because the motion of the particle at each instant is got by compounding its previous motion with that due to the central force.

We shall accordingly take this plane, called the *plane of the orbit*, as the plane of rectangular coordinate axes; the fixed centre of force being the origin O.

151. **Differential Equations of Motion.**—Suppose the force attractive, and P the position of the attracted particle at the end of any time t.



Let

$ON=x$, $PN=y$, $OP=r$, $\angle XOP=\theta$.

Suppose F to represent the *acceleration* due to the attractive force; then, by Art. 68, we have

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= -F\cos\theta = -F\frac{x}{r}\\ \frac{d^2y}{dt^2} &= -F\sin\theta = -F\frac{y}{r}\end{aligned}\right\}. \qquad (1)$$

The complete determination of the motion for any law of force depends on the solution of these simultaneous equations.

In the case of a *repulsive* force it is necessary to change the sign of F.

The path described is evidently always *concave* to the centre of force for *attractive* forces, and *convex* for *repulsive*.

152. **Law of Direct Distance.**—There is one case in which the differential equations can be immediately integrated, viz., when the force varies directly as the distance from the fixed centre.

Let $F = \mu r$; then, for attractive forces, we have

$$\left.\begin{aligned}\frac{d^2x}{dt^2} + \mu x &= 0\\ \frac{d^2y}{dt^2} + \mu y &= 0\end{aligned}\right\}. \qquad (2)$$

The integrals of these equations, by Art. 109, may be written

$$\left.\begin{aligned}x &= A\cos t\sqrt{\mu} + B\sin t\sqrt{\mu}\\ y &= A'\cos t\sqrt{\mu} + B'\sin t\sqrt{\mu}\end{aligned}\right\}. \qquad (3)$$

The arbitrary constants in this, as in all other cases, can be found from knowing the position, velocity, and direction of motion at the first instant.

153. **Equation of Orbit, and Periodic Time.**—If we solve the preceding equations for $\cos t\sqrt{\mu}$ and $\sin t\sqrt{\mu}$, and add the squares of the results, we get

$$(Ay - A'x)^2 + (By - B'x)^2 = (AB' - BA')^2 \qquad (4)$$

This equation represents an ellipse, whose centre is at the centre of force.

Again, if $2\pi + t\sqrt{\mu}$ be substituted for $t\sqrt{\mu}$ in equations (3), the values of x and y remain unaltered; hence, if $(t' - t)\sqrt{\mu} = 2\pi$, the body will occupy the same position at the end of the time t' which it occupied at the time t. Accordingly, if T be the time of a complete revolution in the orbit, we have

$$T = \frac{2\pi}{\sqrt{\mu}}.$$

T is called the *periodic* time, and is the same for all orbits round the same centre of force, since it depends only on μ, the intensity of the central force, *i.e.* the acceleration at the unit of distance, and not on the initial conditions of the motion.

154. **Determination of the Arbitrary Constants.**—Let a, b be the coordinates of the particle at the instant from which the time is reckoned, V the initial velocity, and α the angle which the initial direction of motion makes with the axis of x; then, making $t = 0$ in equations (3), we get $A = a$, $A' = b$.

Again, by differentiation, we have

$$\frac{dx}{dt} = B\sqrt{\mu}\cos t\sqrt{\mu} - A\sqrt{\mu}\sin t\sqrt{\mu},$$

$$\frac{dy}{dt} = B'\sqrt{\mu}\cos t\sqrt{\mu} - A'\sqrt{\mu}\sin t\sqrt{\mu}.$$

Hence $\qquad V\cos\alpha = B\sqrt{\mu}, \quad V\sin\alpha = B'\sqrt{\mu};$

consequently,
$$\left.\begin{aligned} x &= a\cos t\sqrt{\mu} + \frac{V\cos\alpha}{\sqrt{\mu}}\sin t\sqrt{\mu} \\ y &= b\cos t\sqrt{\mu} + \frac{V\sin\alpha}{\sqrt{\mu}}\sin t\sqrt{\mu} \end{aligned}\right\}; \quad (5)$$

thus the position of the particle at any instant is determined.

155. **Repulsive Force.**—Next, if the force be repulsive the equations of motion are

$$\frac{d^2x}{dt^2} = \mu x, \quad \frac{d^2y}{dt^2} = \mu y.$$

Hence, as before,
$$\left.\begin{array}{l} x = Ae^{t\sqrt{\mu}} + Be^{-t\sqrt{\mu}} \\ y = A'e^{t\sqrt{\mu}} + B'e^{-t\sqrt{\mu}} \end{array}\right\}. \tag{6}$$

If we solve for $e^{t\sqrt{\mu}}$ and $e^{-t\sqrt{\mu}}$, and multiply the resulting values, we get
$$(A'x - Ay)(By - B'x) = (A'B - B'A)^2. \tag{7}$$
This represents a hyperbola, having the lines
$$A'x - Ay = 0, \quad By - B'x = 0$$
for its asymptotes. The constants A, B, A', B' can be easily determined, as in the former case, whenever the initial position, velocity, and direction of motion are given.

Conversely to the preceding Articles, it can be readily shown that if a particle describe a conic under the action of a force directed to its centre, the force varies directly as the distance; and is attractive for an ellipse, and repulsive for a hyperbola.

156. **Several Centres of Force.**—The results arrived at above hold for the motion of a body acted on by any number of centres of force, each varying directly as the distance. For it is readily seen that, in this case, the forces are equivalent to a single force, directed to the centre of mean position of the different centres of force, whose intensity or *absolute* force is equal to the sum of the intensities of the different centres of force (see Minchin's *Statics*, Art. 23).

In like manner, if we suppose each particle of a body to attract according to the law of direct distance, its total attraction is the same as if its entire mass were concentrated at its centre of inertia.

Hence it follows that if two bodies mutually attract, according to this law, their centres of inertia will describe ellipses, in the same periodic time, round their common centre of inertia. This result holds good for any number of mutually attracting bodies. In all cases the path described by the centre of inertia of a body is called the orbit of the body.

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Examples.

1. Prove that the velocity at any point in a central elliptic orbit varies directly as the diameter drawn parallel to the tangent at the point.

2. In the case of a repulsive force, varying as the distance, find the arbitrary constants, the initial conditions being supposed the same as in Art. 154.

Making $t = 0$ in equations (6), we get $a = A + B$, $b = A' + B'$.

Again, by differentiation, on making $t = 0$, we get

$$V \cos \alpha = (A - B)\sqrt{\mu}, \quad V \sin \alpha = (A' - B')\sqrt{\mu}.$$

Hence,

$$A = \frac{1}{2}\left(a + \frac{V \cos \alpha}{\sqrt{\mu}}\right), \quad B = \frac{1}{2}\left(a - \frac{V \cos \alpha}{\sqrt{\mu}}\right),$$

$$A' = \frac{1}{2}\left(b + \frac{V \sin \alpha}{\sqrt{\mu}}\right), \quad B' = \frac{1}{2}\left(b - \frac{V \sin \alpha}{\sqrt{\mu}}\right).$$

3. Find the condition that the orbit in the preceding should be an equilateral hyperbola. *Ans.* $V^2 = (a^2 + b^2)\mu$.

4. A body is acted on by four equal masses, attracting directly as the distance; find its orbit, and show that its periodic time is one-half of that of a body acted on by one of the masses alone.

5. A body is attracted to one fixed centre, and repelled by another, of equal intensity, each varying directly as the distance. Find its path.

Ans. A parabola.

6. In the ellipse described freely by a body, under the action of a central force varying directly as the distance, determine the relation connecting the eccentric angle of position with the time of passage through any point on the curve.

7. A number of bodies, which describe ellipses about the centre of force as centre in the same periodic time, are projected from a given point with a given velocity in different directions in a plane. Prove that their paths will all touch a fixed ellipse with the given point as focus. *Camb. Math. Trip.*, 1875.

8. Being given the centre of force, a point in the orbit, and the velocity and direction of motion at that point; give a geometrical construction for the lengths and positions of the axes major and minor of the orbit.

We now return to the general equations of motion under

CENTRAL FORCES.

157. Equable Description of Areas.—In equations (1) if the first be multiplied by y, and the second by x, we get by subtraction

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0, \text{ or } \frac{d}{dt}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = 0.$$

Hence

$$x\frac{dy}{dt} - y\frac{dx}{dt} = h,$$

where h is a constant independent of the time.

Again (Art. 105, *Diff. Calc.*), we have

$$x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dt};$$

therefore
$$r^2\frac{d\theta}{dt} = h. \tag{8}$$

Hence, if A denote the area described in the time t by the radius vector r drawn to the particle, we have

$$\frac{dA}{dt} = \tfrac{1}{2}\left(r^2\frac{d\theta}{dt}\right) = \tfrac{1}{2}h;$$

therefore
$$A = \tfrac{1}{2}(ht). \tag{9}$$

No constant is added since we suppose A and t to vanish together.

If we suppose $t = 1$, we infer that h is *double the area described by the radius vector in the unit of time.*

Conversely, if a particle move in a plane, and describe equal areas in equal times around a fixed point in the plane, then the entire force acting on it at each instant passes through the fixed point (compare Art. 28).

158. **Velocity at any Point.**—Again (Art. 183, *Diff. Calc.*), we have

$$p\frac{ds}{dt} = r^2\frac{d\theta}{dt},$$

where ds denotes the element of the path described in the time dt, and p is the length of the perpendicular from the centre of force on the tangent at the point. Hence

$$p\frac{ds}{dt} = h; \text{ but } \frac{ds}{dt} = v,$$

where v denotes the velocity at the instant; therefore

$$v = \frac{h}{p}. \tag{10}$$

Accordingly the velocity varies inversely as the perpendicular p.

The constant h can be determined from (10) whenever the velocity V, the distance R, and the direction of motion at any point of the path of the particle, are known.

For, let ϕ denote the angle which the direction of motion, at the instant, makes with the radius vector R; then the perpendicular on the tangent $= R \sin \phi$, and hence

$$h = VR \sin \phi. \tag{11}$$

Equation (10) admits of another form; for, squaring, it becomes

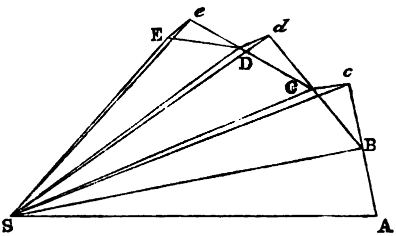
$$v^2 = \frac{h^2}{p^2};$$

therefore
$$v^2 = h^2\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\},$$

where $u = \dfrac{1}{r}$ (*Diff. Calc.*, Art. 183). (12)

159. **Newton's Proof.**—On account of the importance of the preceding results we shall give the method by which the equable description of areas was originally established by Newton.

Let the whole time be divided into a number of equal intervals. Then, supposing no force to act on the body during the first interval, it would describe a right line AB, uniformly, in that interval. Likewise during the next interval, if no force act on it, it would describe the right line Bc, in the direction of, and equal to, AB. But when the body arrives at B, suppose a force directed to S to act on it, with a *single sudden and great impulse*, so as to cause the body to deviate from the right line Bc, and to proceed along the line BC. To find the position of the body at the end of the second interval, we draw from c the line cC parallel to BS (the direction of the force), and meeting BC in C; then the body will be found at C at the end of this interval. Join SC and Sc; then, since



SB and Cc are parallel, the triangle SBC is equal to SBc, and therefore equal to the triangle SAB. In like manner D, E, &c. the positions at the end of the next intervals, can be determined. Also it is obvious that the right lines AB, BC, CD, &c., all lie in the same plane, and the triangles SCD, SDE, &c., will be each equal to SAB.

Therefore equal areas round S are described in equal intervals of time; and, *componendo*, the sum of the areas described are proportional to the time of their description.

If now we suppose the number of intervals of time increased, and their length diminished indefinitely, the path described becomes a curved line; the centripetal force by which the body is perpetually deflected from the tangent to the curve will act continuously; and the areas described round S, being always proportional to the time of their description, will be so in this case also.

The other results of the preceding Article follow likewise (Newton, Lib. I., Sec. II., Prop. i., *Principia*).

160. **Velocity at any Distance.**—In equations (1) if we multiply the first by $2dx$, and the second by $2dy$, and add, we get

$$2\frac{d^2x}{dt^2}\,dx + 2\frac{d^2y}{dt^2}\,dy = -2F\frac{xdx + ydy}{r} = -2Fdr.$$

Integrating, we get

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = -2\int Fdr + \text{const.},$$

or

$$v^2 = -2\int Fdr + \text{const.} \tag{13}$$

By aid of this equation, when the law of force is given, the velocity at any point in the orbit can be determined.

Thus, let the acceleration F be any function of the distance represented by $\mu\phi'(r)$, then

$$v^2 = -2\mu\int\phi'(r)\,dr + \text{const.} = -2\mu\phi(r) + \text{const.}$$

Again, let V be the velocity at the distance R, and we get

$$V^2 = -2\mu\phi(R) + \text{const.};$$

therefore

$$v^2 - V^2 = 2\mu\{\phi(R) - \phi(r)\}. \tag{14}$$

For instance, for the law of nature, we have

$$v^2 - V^2 = 2\mu\left(\frac{1}{r} - \frac{1}{R}\right). \tag{15}$$

Hence we see that the velocity at any distance from the centre of force is independent of the path described, and is the same as if the body had been projected, with the initial velocity, directly towards the centre of force (compare Art. 131).

Again, if $f = \frac{\mu}{r^n}$, we have

$$v^2 - V^2 = \frac{2\mu}{n-1}\left(\frac{1}{r^{n-1}} - \frac{1}{R^{n-1}}\right). \tag{16}$$

If $V = 0$, when $R = \infty$, *i.e.* if *the velocity at any point in the path is that which the body would acquire in moving from rest from an infinitely great distance* towards the centre of force, we have

$$v^2 = \frac{2\mu}{n-1}\frac{1}{r^{n-1}}. \tag{17}$$

For instance, if the force vary as the inverse square of the distance, we have in this case

$$v^2 = \frac{2\mu}{r}. \tag{18}$$

Again, if the force be *repulsive*, and vary directly as the n^{th} power of the distance, we have $F = -\mu r^n$, and (14) becomes

$$v^2 - V^2 = \frac{2\mu}{n+1}(r^{n+1} - R^{n+1}). \tag{19}$$

If $V = 0$ when $R = 0$, *i.e.* if *the velocity at any point be the same as that acquired in moving from the centre of force,*

$$v^2 = \frac{2\mu}{n+1} r^{n+1}. \tag{20}$$

161. **Law of Inverse Square.**—If $F = \frac{\mu}{r^2}$, equations (1) become

$$\ddot{x} = -\frac{\mu x}{r^3}, \quad \ddot{y} = -\mu\frac{y}{r^3}. \tag{21}$$

Also from (8), we have, $\quad \frac{1}{r^2} = \frac{\dot{\theta}}{h}$;

hence, equations (21) become

$$\ddot{x} = -\frac{\mu}{h}\frac{x}{r}\,\dot{\theta} = -\frac{\mu}{h}\cos\theta\,\dot{\theta}.$$

$$\ddot{y} = -\frac{\mu}{h}\frac{y}{r}\,\dot{\theta} = -\frac{\mu}{h}\sin\theta\,\dot{\theta}.$$

Integrating, we get,

$$\left.\begin{aligned} \dot{x} &= -\frac{\mu}{h}\sin\theta + \alpha \\ \dot{y} &= \frac{\mu}{h}\cos\theta + \beta \end{aligned}\right\}, \tag{22}$$

in which α and β are constants, whose values can be found by the aid of the initial circumstances of the motion.

Again, substituting these values of $\dot{x}$ and $\dot{y}$ in the equation $x\dot{y} - y\dot{x} = h$, we get

$$\frac{\mu}{h}r + \beta x - \alpha y - h = 0. \tag{23}$$

From this it follows that the orbit is a conic section having the centre of force at its focus.

Further discussion of this law of force is postponed to Art. 166, in which will be given another demonstration that the orbit is a focal conic.

162. **To prove the Relation** $F = \frac{h^2}{p^3}\frac{dp}{dr}$.

Equation (13) gives, by differentiation,

$$F = -\frac{1}{2}\frac{d(v^2)}{dr} = -\frac{1}{2}h^2\frac{d}{dr}\left(\frac{1}{p^2}\right) = \frac{h^2}{p^3}\frac{dp}{dr}. \tag{24}$$

This result admits of a useful transformation; for, if γ denote the semichord of curvature drawn through the centre of force, we have

$$\gamma = p\frac{dr}{dp}. \quad (\textit{Diff. Calc.}, \text{Art. } 235.)$$

Hence the previous equation becomes

$$F = \frac{v^2}{\gamma}. \tag{25}$$

This result can also be readily deduced from the consideration that the centrifugal acceleration, $\frac{v^2}{\rho}$, at any point in the orbit, must be equal and opposite to the component of the central acceleration taken in the normal direction (Arts. 25, 90).

Examples.

1. Prove that the velocity at any point in a central orbit is the same as that acquired in moving from rest along one-fourth the chord of curvature at the point, under the action of a constant force, equal in intensity to that of the central force at the point.

2. A particle describes a circle freely under the action of a force whose direction is constant; determine the law of force.

Taking the centre of the circle as origin of rectangular axes, the axis of y being parallel to the constant direction of the force, we have

$$\frac{d^2y}{dt^2} = Y, \quad \frac{d^2x}{dt^2} = 0, \quad x^2 + y^2 = a^2;$$

$$\therefore \quad \frac{dx}{dt} = a, \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0;$$

hence,
$$\frac{dy}{dt} = -a\frac{x}{y};$$

hence,
$$Y = \frac{d^2y}{dt^2} = \frac{a}{y^2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = -\frac{a^2a^2}{y^3}.$$

3. Apply equation (24) to find the law of force directed to a focus in an ellipse.

In this case we have

$$\frac{1}{p^2} = \frac{1}{b^2}\left(\frac{2a}{r} - 1\right);$$

$$\therefore \quad \frac{1}{p^3}\frac{dp}{dr} = \frac{a}{b^2}\frac{1}{r^2}; \quad \text{hence,} \quad F = \frac{ah^2}{b^2r^2}.$$

4. **Find the law of force in the curve**

$$r^m = a^m \cos m\omega.$$

Here we have (*Diff. Calc.*, Art. 190) $r^{m+1} = a^m p$.

Hence,
$$F = \frac{(m+1)\,h^2 a^{2m}}{r^{2m+3}}.$$

5. Prove that the force under whose action a body P revolves in any orbit, about a centre of force S, is to the force under whose action the same body P can revolve in the same orbit, in the same time, round another centre of force R, as $SP \,.\, RP^2 : SG^3$, where SG is the straight line drawn from S parallel to RP, meeting in G the tangent at P to the orbit. *Principia*, Sect. II., Prop. VII., *Cor.* 3.

163. To prove the Equation $\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}$.

In the equation

$$\frac{d(v^2)}{dr} = -2F,$$

if we regard r as a function of θ, we have

$$-2F = \frac{\frac{d(v^2)}{d\theta}}{\frac{dr}{d\theta}} = -\frac{u^2}{\frac{du}{d\theta}}\frac{d(v^2)}{d\theta}.$$

Moreover, from (12), we have

$$\frac{d(v^2)}{d\theta} = 2h^2\frac{du}{d\theta}\left(u + \frac{d^2u}{d\theta^2}\right).$$

Substituting in the preceding, we get

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}. \qquad (26)$$

This important result can also be proved as follows:— Substituting $-F$ for P, in equation (11), Art. 28, we get

$$F = -\frac{d^2r}{dt^2} + r\left(\frac{d\theta}{dt}\right)^2;$$

but
$$r\left(\frac{d\theta}{dt}\right)^2 = h^2u^3, \text{ by (8)};$$

also
$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = hu^2\frac{dr}{d\theta} = -h\frac{du}{d\theta};$$

$$\therefore \quad \frac{d^2r}{dt^2} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right) = -h\frac{d\theta}{dt}\frac{d^2u}{d\theta^2} = -h^2u^2\frac{d^2u}{d\theta^2};$$

consequently
$$F = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right).$$

The discussion of central orbits comprises two distinct classes of questions. In the one it is required to find the equation of the orbit when the law of force is known; in the other the orbit described is given, and the law of force, directed to a fixed point, is required.

In the latter case, if the origin be taken at the fixed centre of force, the equation of the orbit can, in general, be expressed in terms of u and θ, from which the value of $\frac{d^2u}{d\theta^2}$ can be determined. If this be substituted in the equation

$$F = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right),$$

the resulting value of F determines the required law of force.

164. **Application to Ellipse.**—For example, to find the law of force which will cause a particle to describe an ellipse round a centre of force situated in one of its foci.

Here the equation of the orbit is

$$u = \frac{1 + e\cos\theta}{L},$$

where L is the semi latus-rectum.

Hence
$$\frac{d^2u}{d\theta^2} = -\frac{e\cos\theta}{L};$$

therefore
$$u + \frac{d^2u}{d\theta^2} = \frac{1}{L},$$

and consequently
$$F = \frac{h^2u^2}{L} = \frac{h^2}{a(1-e^2)}\frac{1}{r^2}. \qquad (27)$$

Accordingly the force varies inversely as the square of the distance from the centre of force.

Examples.

Find the law of force, directed to the origin, in the following curves :—

1. $r = ae^{a\theta}$. 2. $u = ae^{a\theta} + be^{-a\theta}$. 3. $r = ae^{a\theta} + be^{-a\theta}$.

Ans. 1. and 2. $\frac{1}{r^3}$. 3. $F = \frac{\mu}{r^2}\left(\frac{1+a^2}{r} - \frac{8a^2ab}{r^3}\right)$.

165. **Case where the Law of Force is given.**— When the law of force is given, the determination of the orbit depends on the solution of a differential equation; for, if $F = \mu\phi(u)$, equation (26) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}\frac{\phi(u)}{u^2} \qquad (28)$$

This equation admits of being completely integrated for a few laws of force only. We shall commence with the most important case, namely, the law of nature, for which the attraction varies as the inverse square of the distance.

166. **Law of Inverse Square.**—Let $F = \frac{\mu}{r^2} = \mu u^2$, then the equation becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}.$$

The integral of this, by Art. 109, is

$$u = \frac{\mu}{h^2} + A \cos(\theta - \alpha). \tag{29}$$

This is the equation of a focal conic (see Art. 161).

The orbit is an ellipse, parabola, or hyperbola, according to the values of the constants A and α. These constants are, as in all other cases, determined from the initial circumstances of the motion.

We commence with the case in which the orbit is an ellipse.

The equation of an ellipse referred to a focus as origin, and to any line drawn through it as prime vector, may be written

$$u = \frac{1}{r} = \frac{1 + e\cos(\theta - \alpha)}{a(1 - e^2)}.$$

Comparing with (29) we get

$$\frac{\mu}{h^2} = \frac{1}{a(1 - e^2)} = \frac{a}{b^2},$$

or

$$h^2 = \mu \frac{b^2}{a} = \mu L. \tag{30}$$

Hence, in different orbits round the same centre of force, h varies as the square root of the latus rectum.

Again, let T denote the *periodic time*, i.e. the time in which the body makes a complete revolution in the orbit; then since h represents double the area described in the unit of time, we have

$$h = \frac{\text{double area of ellipse}}{T} = \frac{2\pi ab}{T}.$$

Hence, from (30),

$$\mu = \frac{4\pi^2 a^3}{T^2}. \tag{31}$$

If a second particle be supposed to describe an ellipse round the centre of force, and if the *absolute force* μ be the same in both cases, we have

$$\mu = \frac{4\pi^2 a'^3}{T'^2},$$

where a', T' are the semi-axis and the periodic time in its orbit. Hence, eliminating μ, we get

$$\left(\frac{T}{T'}\right)^2 = \left(\frac{a}{a'}\right)^3. \qquad (32)$$

That is, *the squares of the periodic times are to one another in the same ratio as the cubes of the semi-axes major.*

167. The preceding results have been deduced for the motion of a material particle, but they also hold good, approximately, for the motion of the centre of inertia of a body of finite dimensions, each of whose elements is attracted towards a fixed centre, provided the dimensions of the body are small in comparison with its distance from the centre of force. For in this case the attractions on the several elementary particles of the body may be, approximately, regarded as a system of equal and parallel accelerations; and, consequently, the motion of the body will (Art. 34) be the same as if it were concentrated at its *centre of inertia.* Also, as already shown in Art. 126, if a sphere consist of homogeneous spherical strata, its entire attraction is the same as if its entire mass were concentrated at its centre. Accordingly, if one such sphere be attracted by another supposed at rest, its centre will describe an ellipse, having the centre of the attracted sphere for a focus.

168. **Kepler's Laws.**—By comparing the results of a large series of observations of the planets, chiefly of Mars, made by Tycho Brahe, Kepler arrived at the following laws concerning the planetary orbits:—

(1) That the right line drawn from the Sun to any planet describes equal areas in equal times.

(2) That the orbits are ellipses, having the Sun in a focus.

(3) That the squares of the periodic times for any two planets are to each other in the same proportion as the cubes of their mean distances from the Sun.

From the first of these laws Newton deduced (Art. 157) that each of the planets is kept in its orbit by the action of a central force directed to the Sun.

From the second he proved that the attractive force for each planet, in its different positions, varies as the inverse square of the distance from the Sun (Art. 164).

From the third law he deduced that the absolute force (μ) is the same for all the planets (Art. 166); and hence that it is one and the same force, directed to the Sun, by which all the planets are retained in their orbits. These laws are only approximate when we take account of the mutual actions of the planets on each other and on the Sun.

For Newton's demonstrations the student is referred to the *Principia*, Lib. I., Sect. III., Prop. XI.

From the foregoing we infer that the results arrived at for the motion of a particle, for the law of inverse square of the distance, are applicable, approximately, to the planetary motions. It has also been verified by observation that a satellite belonging to any planet revolves round it according to the same laws that the planets revolve round the Sun.

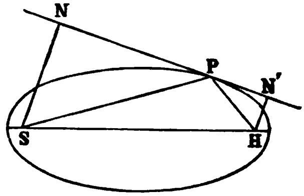
169. **Law of Gravitation.**—We have in the last Article given a brief outline of the process by which Newton established the great fundamental law of attraction of matter, which we refer to as the law of nature, and which may be stated as follows:—*Every particle of matter in the solar system, consisting of the Sun, the planets, comets, &c., exercises on every other particle an attractive force, which varies directly as the product of the masses of the particles, and inversely as the square of their mutual distance.*

We assume that this is a general property of matter, and applies to all matter wherever existing in the universe. This assumption has been verified by observations on the motion of the double stars.

170. Expression for Velocity at any Point in a Focal Orbit.—We commence with an elliptic orbit.

In this case we have

$$v^2 = \frac{h^2}{p^2} = \frac{\mu}{a}\frac{b^2}{p^2}, \text{ by (30)},$$



where $p = SN$,

the centre of force being S.

Again, let H be the second focus of the orbit, SN, HN' perpendiculars on the tangent at P, the position of the particle.

Suppose $HN' = p'$, $HP = r'$; then, from well-known elementary properties of the ellipse, we have

$$r + r' = 2a, \quad pp' = b^2, \quad \frac{p'}{p} = \frac{r'}{r}.$$

Hence
$$v^2 = \frac{\mu}{a}\frac{pp'}{p^2} = \frac{\mu}{a}\frac{p'}{p} = \frac{\mu}{a}\frac{r'}{r} = \frac{\mu}{a}\left(\frac{2a - r}{r}\right);$$

therefore
$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a}. \qquad (33)$$

In the parabola a becomes infinite, and we have

$$v^2 = \frac{2\mu}{r}, \qquad (34)$$

a result which can be readily established independently.

In the case of a hyperbolic path we have $r' = 2a + r$, and the formula becomes

$$v^2 = \frac{2\mu}{r} + \frac{\mu}{a}. \qquad (35)$$

Hence we infer that if a body be projected with a velocity V, at a distance R from the centre of force, the orbit described will be an ellipse, parabola, or hyperbola, according as

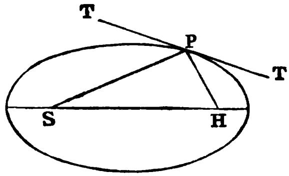
$$V^2 \text{ is} < = \text{ or} > \frac{2\mu}{R}.$$

This result may be exhibited in another form by aid of equation (18), as follows:—

The velocity at any point in an ellipse is less, in a para-

bola equal to, and in a hyperbola greater than, the velocity which the body would acquire in moving to the point from an infinitely great distance, under the action of the central force.

171. **Construction of Orbit.**—The preceding equation shows how to construct the orbit when we are given the absolute force, the initial velocity, position, and direction of motion. For, suppose P the initial position, PT the direction of motion, and S the centre of force; let V = velocity of projection, $SP = R$; then—

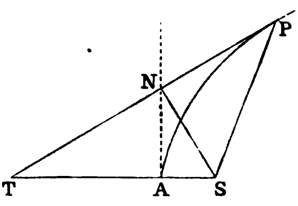


(1) if $V^2 < \frac{2\mu}{R}$ the orbit is an ellipse whose semi-axis a is given by the equation

$$\frac{1}{a} = \frac{2}{R} - \frac{V^2}{\mu}.$$

Again, draw PH, making the angle $T'PH = \angle SPT$, then the second focus H lies on this line, and its position H is found by taking $PH = 2a - R$. Consequently, as the two foci and the axis major are known, the ellipse is completely determined.

(2) When $\frac{2\mu}{R} = V^2$ the orbit is a parabola, which can be easily determined by drawing SN perpendicular to the direction of motion at P, inflecting $ST = SP$, and dropping NA perpendicular to ST.



The parabola described with S for focus, and A for vertex, will be the required orbit.

(3) When $V^2 > \frac{2\mu}{R}$ the orbit is a hyperbola, whose semi-axis a is given by the equation

$$\frac{1}{a} = \frac{V^2}{\mu} - \frac{2}{R}.$$

The second focus, H, can be easily constructed, as in the first case, but lies on the opposite side of the direction of motion from the centre of force S.

Again, as the value of the semi-axis a is independent of the direction of projection, we infer that if a number of bodies be projected from a point with the same velocity, in different directions, and be attracted by a common centre of force, the mean distances, and consequently the periodic times, will be the same for all the orbits.

It may be remarked that the orbit will be a circle, provided the angle SPT is right, and $V^2 = \frac{\mu}{R}$ (compare Art. 91).

The formulæ in this and the preceding Article are of importance in the discussion of focal orbits. We add a few elementary applications.

EXAMPLES.

1. Calculate, approximately, the periodic time of a planet if its mean distance from the Sun is double that of the Earth. *Ans.* 1033 days.

2. If a body be projected with a given velocity about a centre of force which varies as the inverse square of distance, find the locus of the centre of the orbit described.

Here, since the locus of the empty focus is a circle, the locus of the centre is also a circle.

3. In the same case, show that the length of the axis-minor varies directly as the perpendicular drawn from the centre of force to the direction of projection.

Since r and r' are each constant, p is to p' in a constant ratio; consquently b varies as p.

4. Show that there are two directions in which a body may be projected from a given point A, with a given velocity V, so as to pass through another given point B.

Since the axis-major $2a$ is given, the position of the second focus is determined by the intersection of two circles, with A and B for centres. Hence there are two solutions—one for each point of intersection of the circles.

5. Prove that the time of describing an arc of a parabolic orbit, bounded by a focal chord of length c, varies as $c^{\frac{3}{2}}$.

172. Effect of a Sudden Change in Absolute Force.—A body is revolving in a focal orbit; if when it arrives at any position the absolute force μ be suddenly altered, to determine the subsequent path.

Let R and V represent the distance and velocity at the instant in question, and let μ' be the new value of the absolute force, and a' the semi-axis major of the new orbit; then, as the velocity receives no sudden or instantaneous change, we have, by (33),

$$\frac{2\mu}{R} - \frac{\mu}{a} = \frac{2\mu'}{R} - \frac{\mu'}{a'}. \tag{36}$$

The value of a', and consequently the position of the new orbit, can be immediately determined from this equation.

For example, suppose the original orbit a parabola, and the central force suddenly doubled in intensity.

Here $\mu' = 2\mu$, and our equation becomes

$$\frac{2\mu}{R} = \frac{4\mu}{R} - \frac{2\mu}{a'}:$$

hence $a' = R$; and, consequently, the new orbit is an ellipse
/minor having the extremity of its axis ~~major~~ at the point.

If the change in μ be very small, and represented by $\Delta\mu$, and the corresponding change in a by Δa, it is plain that we have

$$\Delta a = -\frac{a^2}{\mu}\,\Delta\mu\left\{\frac{2}{R} - \frac{1}{a}\right\}. \tag{37}$$

Hence, if the central force (or the attracting mass) be increased slightly, the axis major will be diminished; also, if the force be diminished the axis major is increased.

The corresponding change in the periodic time is readily found; for, by (31), we have

$$2\log T + \log\mu = 2\log 2\pi + 3\log a;$$

hence
$$\frac{2\Delta T}{T} = \frac{3\Delta a}{a} - \frac{\Delta\mu}{\mu};$$

therefore
$$\frac{\Delta T}{T} = -\frac{\Delta\mu}{\mu}\left(\frac{3a}{R} - 1\right). \tag{38}$$

Again, if the centre of force be supposed suddenly transferred to a new position, the subsequent path can be readily constructed, as in Art. 171.

EXAMPLES.

1. A number of bodies are projected from a point with the same velocity, but in different directions; prove that the centres of their orbits are situated on the surface of a sphere.

2. A body is describing a circle under a central force in its centre; if the force be suddenly reduced to one-half, find the subsequent path of the body.

Ans. a parabola.

3. In the same case, if the central force be suddenly increased in the ratio of $m : 1$, find the eccentricity of the subsequent path. *Ans.* $\dfrac{m-1}{m}$.

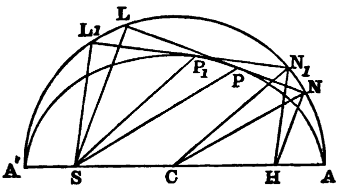
4. Two equal perfectly elastic particles describe the same ellipse in the same period, in opposite directions, one about each focus; prove that the major axis of the orbit is a harmonic mean between those of the orbits they will describe after impact.

This result follows immediately, since the *vis viva* is the same after collision as before (*see* Art. 81).

5. Prove that there are two initial directions for the projection of a particle with a given velocity, so that the axis major of its orbit may coincide in direction with a given line.

6. If, when the Earth is at an end of the minor axis of its elliptic orbit, a meteor were to fall into the Sun, whose mass is the m^{th} part of that of the Sun; find the resulting change in the Earth's mean distance, and also in the length of the year. *Ans.* $\Delta a = -\dfrac{a}{m}$, $\Delta t = -\dfrac{2T}{m}$.

173. Application of Method of Hodograph.—The method of the hodrograph (Art. 26) furnishes a simple mode of determining the law of force in a focal ellipse. For, since the velocity at any point P varies inversely as the perpendicular SL, it varies directly as the perpendicular HN drawn from the second focus; since $SL \times HN = b^2$.



Consequently the hodograph is similar to the locus of N, when turned through a right angle. But the semicircle described on the axis major as diameter passes through N, consequently the hodograph is a circle.

Again, to find the law of force, let P_1 denote the position of the movable at the end of an indefinitely small time Δt,

and N_1 the corresponding position of N; then (Art. 26) $\frac{NN_1}{\Delta t}$ is proportional to the central attractive force.

Join the centre C to N and to N_1; then, by an elementary property of the ellipse, CN is parallel to SP, and CN_1 to SP_1.

Let $SP = r$, $\angle CSP = \theta$, $SL = p$, $HN = p'$;

then $$\angle NCN_1 = \angle PSP_1 = \Delta\theta.$$

Also (by 8), $$\frac{NN_1}{\Delta t} = a\frac{\Delta\theta}{\Delta t} = \frac{ah}{r^2}.$$

Hence the force varies inversely as the square of the distance.

Again, since $v = \frac{h}{p} = \frac{h}{b^2}p'$, we have

$$F = \frac{h}{b^2}\frac{NN_1}{\Delta t} = \frac{h^2 a}{b^2}\cdot\frac{1}{r^3}.$$

Consequently, if μ represent the absolute force, *i.e.* the force at unit of distance, we get

$$\mu = \frac{h^2 a}{b^2},$$

as in (30).

Again, since the velocity at P is proportional and perpendicular to HN; and CN, CH are constants, it follows that the velocity at P can be resolved into two *constant velocities—one perpendicular to the radius vector, the other to the axis major.*

Also, since the velocity at P is represented by $\frac{h}{b^2}HN$, the component velocity perpendicular to SP is represented by $\frac{ha}{b^2}$, and that perpendicular to the axis major by $\frac{hae}{b^2}$: i.e. by $\frac{\mu}{h}$ and $\frac{\mu}{h}e$, or by $\sqrt{\frac{\mu}{L}}$ and $\sqrt{\frac{\mu}{L}}e$, respectively.

That the hodograph is a circle in this case appears also at once from (22). For if x', y' be the coordinates of the

point in the hodograph which corresponds to the point xy in the orbit, we have

$$x' = \dot{x}, \quad y' = \dot{y};$$

hence, substituting in (22), and eliminating θ, we get for the equation of the hodograph

$$(x' - \alpha)^2 + (y' - \beta)^2 = \frac{\mu^2}{h^2},$$

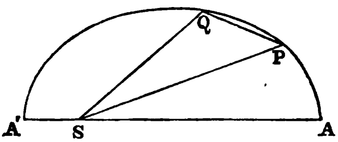
which is the equation of a circle.

We may here observe that in any case of the motion of a particle, if we can find an equation connecting the velocities $\dot{x}$, $\dot{y}$, $\dot{z}$ of the motion, with constants, that equation may be regarded as that of the hodograph, in which $\dot{x}$, $\dot{y}$, $\dot{z}$ are the current coordinates. (See Art. 26.)

Example.

A particle moving in an ellipse under the action of a force directed to a focus has a small velocity $n\frac{\mu}{h}$ impressed on it in the direction of the focus; find the corresponding changes in the eccentricity, and in the position of the apse.

174. **Lambert's Theorem.**—In Art. 140, *Int. Calc.*, it has been shown that the area of the elliptic sector PSQ is represented by



$$\tfrac{1}{2}ab\,\{\phi - \phi' - (\sin\phi - \sin\phi')\},$$

where ϕ and ϕ' are given by the equations

$$\sin\tfrac{1}{2}\phi = \tfrac{1}{2}\left(\frac{r_1 + r_2 + c}{a}\right)^{\frac{1}{2}}, \quad \sin\tfrac{1}{2}\phi' = \tfrac{1}{2}\left(\frac{r_1 + r_2 - c}{a}\right)^{\frac{1}{2}};$$

in which $SP = r_1$, $SQ = r_2$, and $PQ = c$.

Accordingly, if t represent the time of describing the arc PQ, we have

$$t = \frac{2\,\text{area}\,PSQ}{h} = \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}}\{\phi - \phi' - (\sin\phi - \sin\phi')\}. \quad (39)$$

This shows that the time of moving from any point P to any point Q can be expressed in terms of the sides of the triangle SPQ and of the axis major of the orbit.

Again, if we regard a as becoming infinitely great in (39), we get for t, the time of moving from P to Q in a parabolic orbit,

$$t = \frac{1}{6\sqrt{\mu}}\{(r_1 + r_2 + c)^{\frac{3}{2}} - (r_1 + r_2 - c)^{\frac{3}{2}}\}. \tag{40}$$

For in this case we may substitute $\frac{1}{6}\left(\frac{r_1 + r_2 + c}{a}\right)^{\frac{3}{2}}$ for $\phi - \sin\phi$, and $\frac{1}{6}\left(\frac{r_1 + r_2 - c}{a}\right)^{\frac{3}{2}}$ for $\phi' - \sin\phi'$.

Examples.

1. A comet, describing a parabolic orbit, being supposed to cross the path of the Earth; determine the points of ingress and egress for which the time the comet continues within the Earth's orbit is a maximum.

Ans. The extremities of the axis major.

2. Find an expression for the time in the preceding question.

Ans. $\frac{2E}{3\pi}$, where E represents the length of the year.

3. Two planets, describing elliptic orbits in a common period round the Sun, being supposed to pass in every revolution through two common points: prove that the intervals between the times of their passage through the points are equal.

175. **Modification when Mutual Attraction is taken account of.**—The preceding investigations are based on the assumption that the centre of force is fixed; accordingly they can be applied to the planetary motions only on that hypothesis. However, from the principle of the equality of action and reaction, each of the planets exerts on the Sun an equal and opposite attractive force to that which the Sun exerts on it. We proceed to consider how far our results must be modified when this is taken into account.

We have seen, in Art. 13, that the relative motion of two bodies is unaltered if equal and parallel velocities be given to both. We accordingly suppose an acceleration applied at each instant to the Sun, equal and opposite to that which the planet exerts on it; and an equal and parallel acceleration applied to the planet. This assumption will not alter their relative positions, while it reduces the position of the Sun to

one of relative rest. Consequently the *relative motion* of the planet takes place in the same manner as if the Sun were a fixed centre of force, and the planet at each instant were acted on by the sum of the accelerations that the Sun exerts on it, and that it exerts on the Sun; since these accelerations take place in opposite directions along the same right line.

Again, let S and P denote the masses of the Sun and planet respectively: then their attractions (being proportional to their masses) may be represented by $f\dfrac{S}{r^2}$ and $f\dfrac{P}{r^2}$, where r represents their mutual distance.

Accordingly the total acceleration on the planet towards the Sun, considered as a fixed centre, is represented by

$$f\frac{(S+P)}{r^2}.$$

Consequently in our preceding investigations we must regard the absolute force, μ, as proportional to $S+P$ instead of S; and we may, by proper assumption of units, take

$$\mu = f(S+P).$$

176. **Modification in Kepler's Third Law.**—From what has been just established it follows that Kepler's third law is only approximate. To determine a more exact result we must substitute $f(S+P)$ instead of μ, in (31), for one planet, and $f(S+P')$ in the corresponding formula for the other planet, when we have, by division,

$$\frac{S+P}{S+P'} = \left(\frac{a}{a'}\right)^3 \left(\frac{T'}{T}\right)^2. \qquad (41)$$

As observation shows that Kepler's third law is very nearly exact for all the planets, we conclude that the mass of the Sun is very great in comparison with that of any of the planets. In fact the mass of Jupiter, which is the largest of them, is less than a thousandth part of that of the Sun.

This conclusion will appear more clearly from the following method of comparing the mass of the Sun with that of a planet where the planet has a satellite :—

177. **Comparison of Masses of Sun and Planet.**—Let Σ denote the mass of the satellite, δ its distance from the planet, t its periodic time; then, since the satellite revolves round the planet we have, as in last Article,

$$\frac{P+\Sigma}{S+P} = \left(\frac{\delta}{a}\right)^3 \left(\frac{T}{t}\right)^2. \qquad (42)$$

When the calculations are made, it is found that in all cases $\left(\frac{\delta}{a}\right)^3 \left(\frac{T}{t}\right)^2$ is a very small fraction: and hence also $\frac{P}{S}$.

If Σ be supposed very small in comparison with P, as P is in comparison with S, we can, by (42), obtain the ratio of the planet's mass to that of the Sun, approximately.

Again, for two planets, P and P', if the masses of the satellites be neglected, we have

$$\frac{P}{P'} = \left(\frac{\delta}{\delta'}\right)^3 \left(\frac{t'}{t}\right)^2.$$

178. **Mass of Sun.**—When applied to the Earth and its satellite the Moon, the preceding formula gives a means of comparing the mass of the Sun with that of the Earth.

Let E and M represent the masses of the Earth and the Moon, r their distance, then equation (42) becomes

$$\frac{E+M}{S+E} = \left(\frac{r}{a}\right)^3 \left(\frac{T}{t}\right)^2.$$

Now, as a rough approximation, we assume $\frac{r}{a} = \frac{1}{400}$, *i.e.* that the Sun's distance from us is 400 times that of the Moon. Also we take $\frac{T}{t} = 13{\cdot}4$, or that the year is, approximately, 13·4 times the periodic time of the Moon.

This gives $\frac{S+E}{E+M} = \frac{64{,}000{,}000}{179{\cdot}56} = 356{,}420$ approximately.

Moreover, as determined by tidal calculations, $M = \frac{E}{75}$; hence we get

$$\frac{S}{E} = 361{,}473.$$

This result represents very closely the ratio of the Sun's and Earth's mass as determined by more exact investigations.

The foregoing calculation shows the enormous mass of the Sun in comparison with that of the Earth. In like manner the relative masses of Jupiter, Saturn, and other planets which have satellites can be found, approximately.

Examples.

1. Prove that the mass of Jupiter is nearly 270 times the mass of the Earth from the following observations :—Jupiter's fourth satellite is at a mean distance of 25 radii of Jupiter, and its periodic time is 16 days 18 hours; Jupiter's mean radius is 11 times the mean radius of the Earth; the mean distance of the Moon is 60 radii of the Earth, and a mean lunation is 28 days.

2. Prove that the mean density of Jupiter is a little greater than that of water, and that the mean value of g on the surface of Jupiter is about 71, taking the mean density of the Earth as 5·67.

179. **Mean Density of Sun.**—The ratio of the mean density of the Sun to that of the Earth can be determined, as follows:—

From (42) we have, approximately,

$$\frac{S}{E} = \left(\frac{a}{r}\right)^3 \left(\frac{t}{T}\right)^2.$$

Again, let ρ, ρ_1 denote the radii of the Sun and Earth, and σ the ratio of their mean densities; then, assuming them spherical bodies, we have

$$\frac{S}{E} = \sigma \left(\frac{\rho}{\rho_1}\right)^3.$$

Hence $$\sigma = \left(\frac{a}{\rho}\right)^3 \left(\frac{\rho_1}{r}\right)^3 \left(\frac{t}{T}\right)^2,$$

or $$\sigma = \left(\frac{P}{\alpha}\right)^3 \left(\frac{t}{T}\right)^2, \tag{43}$$

where α denotes the Sun's *mean apparent semi-diameter*, and P the Moon's *mean horizontal parallax*.

If we substitute 16′ for a, and 57′ for P, and take $\frac{T}{t}$ as before, we get $\sigma = 0{\cdot}23$, *i.e.* the Sun's mean density is about one-fourth that of the Earth.

It should be observed that this result does not require a knowledge of the Sun's distance; and, as the constants in (43) can be obtained with great accuracy, the ratio of the mean densities of the Sun and Earth can be determined with great precision.

180. **Planetary Perturbations.**—The previous deductions respecting the planetary motions are only approximate for another and a more important reason, namely, that in them we have neglected the mutual actions of the planets on each other.

However, since the Sun's mass is very great in comparison with that of all of the planets, their attractions on any member of the solar system may be regarded as small *disturbing forces*, and the planetary orbits as approximately ellipses.

The usual method of treatment, accordingly, is to regard each planet as moving in an ellipse, in which the elements* are subject to very slow changes, arising from the *perturbations* or disturbing effects of the other planets.

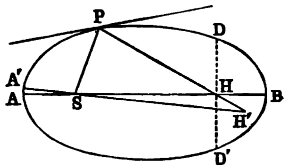
In this manner the problem has been discussed by Lagrange, Laplace, and other great writers on Physical Astronomy. We shall not enter into this discussion, as it is beyond the limits contemplated in this treatise. There is, however, one mode of considering the effects of a disturbing force, which may be here introduced. This consists in supposing the disturbing force resolved into two components†,

* The elements by which a planet's path is determined are—(1) its mean distance from the Sun; (2) its eccentricity; (3) the longitude of its perihelion; (4) the inclination of its plane to a fixed plane; (5) the angle which the intersection of these planes makes with a fixed line; (6) its epoch, or the instant of the planet's being in perihelion.

† There is in general a third component, perpendicular to the plane of the orbit. It is not proposed to consider the effects of this component here. This method of treating the disturbing forces is discussed in a masterly and lucid manner by Sir John Herschel, in his *Outlines of Astronomy*, ch. 12 and 13.

one along the tangent, the other along the normal to the orbit, and in treating their effects separately.

181. **Tangential Disturbing Force.**—Suppose P the position of a planet, moving in the ellipse BPA, in which S and H are the foci; then, since a tangential disturbing force alters the velocity, but produces no effect on the direction of motion, it is easy to find the corresponding changes in the *elements* of the path. For the new position, H', of the second focus will still lie on the line PH.



Again, if v denote the velocity at P, we have, as before,

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a}.$$

When the change in v^2, caused by the tangential disturbing force, is known, the corresponding change in a can be found; and hence the position of H', and consequently that of the new axis major.

Thus if δv be the small change in v, due to the disturbing force, we have

$$2v\delta v = \frac{\mu\delta a}{a^2};$$

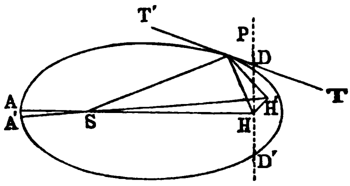
$$\therefore \quad \delta a = \frac{2a^2}{\mu} v\delta v; \qquad \therefore \quad HH' = 2\delta a = \frac{4a^2}{\mu} v\delta v. \quad (44)$$

If the tangential force act in the direction of the motion, and consequently increase the velocity, a will also be increased, and the perihelion A' will consequently move towards P.

Again, the eccentricity e will be increased when SH' is greater than SH, i.e. when P is between the perihelion A and the extremity of the *latus-rectum* drawn through H.

182. **Normal Disturbing Force.**—Next, if a normal

disturbing force act at P, inwardly, it does not alter the velocity, but it changes the direction of motion, through a small angle $\delta\phi$. As the velocity is unchanged, the length of the semi-axis major a is unaltered, while the angle SPT is altered by the quantity $\delta\phi$. Therefore the angle HPH', between PH and the corresponding line PH' in the new orbit, is $2\delta\phi$; also $PH = PH'$. In this manner the position of H' is found when the angle $\delta\phi$ is known. Again, join SH', and produce it at both ends, then the line $A'H'$ will represent the direction of the axis major of the new orbit.



Through H draw DD' perpendicular to SH. The points D and D' are called the *quadratures* of the orbit. When P lies between D and the perihelion A, the line AB, called *the line of apsides* (*see* next Article), moves in the same direction as the planet, and is said to *advance*. The eccentricity increases at the same time. If the planet be between aphelion B and D, the eccentricity continues to increase, and the line of apsides recedes.

Again, in moving from A to D', the *disturbing force still acting inwards*, it is easily seen that the line of apsides *advances*, and the eccentricity *diminishes*. Hence, in the motion from quadrature to quadrature, through perihelion, the apse continually advances, in the case of a normal *disturbing* force acting inwards; the eccentricity increases during the first half of the motion, and diminishes during the second.

The contrary effects have place for a normal disturbing force acting *outwards*.

In like manner in the motion from quadrature to quadrature through *aphelion*, the apse recedes; the eccentricity increases during the first half and diminishes during the second.

183. **Apsides.**—A position for which the moving body is at a maximum or a minimum distance from the centre of force is called an *apse*. The corresponding distance from the centre of force is called an *apsidal distance*, and the line joining the centre of force to an apse is called an *apsidal line*.

Since r, and consequently u, attains a maximum or a minimum value at an apse, we have at such a point

$$\frac{du}{d\theta} = 0.$$

It is easily seen that the orbit is symmetrical at both sides of an apse, provided the force is a function of the distance only. For, if a particle be supposed projected from a point A in a direction perpendicular to the line OA drawn to the centre of force, it is obvious that for the same velocity of projection we must have exactly similar paths, whether it be projected in any given direction or in that exactly opposite. Moreover, if the velocity were reversed at any point, the body would proceed to describe the same orbit, but in an opposite direction. From these considerations it follows that the central orbit must be symmetrical at both sides of an apse, since at that point the motion is perpendicular to the central radius vector.

184. **An Orbit can have but Two Apsidal Distances.**—For, suppose A and B to be two apsides, and the body to move from A to B; then after passing B it will, by the preceding Article, describe a curve similar to BA; and so on. Hence the apsides are constantly repeated, and the angle between two consecutive apsidal distances is the same for all positions of the orbit. This angle is called the *apsidal angle* of the orbit. It is plain that a central orbit cannot be a closed curve unless the apsidal angle is commensurable with a right angle.

185. **Equation for Determination of Apsides.**— Let $F = \mu\phi(u)$, then we have, by (13),

$$v^2 = 2\mu \int \frac{\phi(u)}{u^2}\, du + C,$$

where the value of C is determined by the initial conditions;

$$\text{therefore} \qquad h^2\left(u^2 + \left(\frac{du}{d\theta}\right)^2\right) = 2\mu \int \frac{\phi(u)}{u^2}\, du + C. \qquad (45)$$

Hence, as $\frac{du}{d\theta} = 0$ at an apse, the equation for determining the apsidal distances is

$$h^2 u^2 = 2\mu \int \frac{\phi(u)}{u^2} du + C. \tag{46}$$

If we suppose $F = \mu u^n$, equation (45) becomes

$$h^2 \left\{ \left(\frac{du}{d\theta}\right)^2 + u^2 \right\} = \frac{2\mu}{(n-1)} u^{n-1} + C, \tag{47}$$

and the equation for the apsides

$$h^2 u^2 = \frac{2\mu}{n-1} u^{n-1} + C. \tag{48}$$

The form of the latter equation shows that it cannot have more than two positive roots, which therefore correspond to the two apsidal distances.

For example, let the force consist of two parts, one varying as the inverse square of the distance, the other as the inverse cube, or

$$F = \mu u^2 + \mu' u^3, \tag{49}$$

then $$h^2 u^2 = 2\mu u + \mu' u^2 + C.$$

Accordingly the apsidal distances are in this case determined by a quadratic equation. If $\mu = 0$, there is but one apsidal distance.

186. **Case of Velocity due to an Infinite Distance.**—The integration of equation (47) in a finite form is in general impossible; there is, however, one case in which the equation of the orbit can be readily determined, viz., when the velocity at any point is that acquired in moving from an infinite distance under the action of the central force.

For we have, in this case, by (17), $v^2 = \frac{2\mu}{n-1} u^{n-1}$;

therefore $$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{2\mu}{(n-1) h^2} u^{n-1}. \tag{50}$$

Hence $\dfrac{du}{d\theta} = u\sqrt{au^{n-3} - 1}$, writing a instead of $\dfrac{2\mu}{(n-1)\,h^2}$;

therefore $$\theta = \int \frac{du}{u\sqrt{au^{n-3} - 1}}.$$

To integrate this, let $au^{n-3} = \dfrac{1}{z^2}$, then $\dfrac{du}{u} = -\dfrac{2}{n-3}\dfrac{dz}{z}$;

and we get $$\int \frac{du}{u\sqrt{au^{n-3} - 1}} = -\frac{2}{n-3}\int \frac{dz}{\sqrt{1 - z^2}}$$

$$= \frac{2}{n-3}\cos^{-1}z + \text{const.};$$

$$\therefore \quad \theta + \beta = \frac{2}{n-3}\cos^{-1}z, \text{ or } z = \cos\frac{n-3}{2}(\theta + \beta),$$

where β is an arbitrary constant:

hence $$r^{\frac{n-3}{2}} = \frac{1}{h}\sqrt{\frac{2\mu}{n-1}}\cos\frac{n-3}{2}(\theta + \beta). \qquad (51)$$

If a denote the apsidal distance, and θ be measured from the apsidal line, the preceding may be written

$$r^{\frac{n-3}{2}} = a^{\frac{n-3}{3}}\cos\frac{n-3}{2}\theta. \qquad (52)$$

This is the polar equation of the orbit.

For example, when $n = 2$, we get the parabola

$$r^{\frac{1}{2}}\cos\tfrac{1}{2}\theta = a^{\frac{1}{2}}.$$

Again, when $n = 5$, it becomes

$$r = a\cos\theta;$$

a circle having its centre on the circumference.

O

For $n = 7$ we get the lemniscate

$$r^2 = a^2 \cos 2\theta,$$

and so on.

Equation (52) fails when $n = 3$; in this case, however, (50) becomes

$$\frac{du}{d\theta} = u\sqrt{\frac{\mu}{h^2} - 1},$$

which gives $k\theta = \log u + \text{const.}$, where $k = \sqrt{\frac{\mu}{h^2} - 1}$,

or $$u = \beta e^{k\theta}.$$

This is the equation of a logarithmic spiral.

187. **Approximately Circular Orbits.**—If the orbit described round a centre of force be nearly a circle, its equation can be found approximately, as follows:—

Assume $F = \mu u^2 f(u)$, then equation (26) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} f(u).$$

If the orbit were an exact circle we should have

$$u = a, \text{ and } \frac{d^2u}{d\theta^2} = 0;$$

therefore a must satisfy the equation

$$a = \frac{\mu}{h^2} f(a). \qquad (53)$$

When the orbit is approximately circular we may assume $u = a + z$, where z is always very small.

Hence $$\frac{d^2z}{d\theta^2} + a + z = \frac{\mu}{h^2} f(a + z),$$

or $$\frac{d^2z}{d\theta^2} + a + z = \frac{\mu}{h^2} \{f(a) + zf'(a)\}.$$

By (53) this becomes, neglecting z^2 and higher powers of z,

$$\frac{d^2z}{d\theta^2}+z\left(1-\frac{\mu}{h^2}f'(a)\right)=0;$$

or, substituting $\frac{a}{f(a)}$ for $\frac{\mu}{h^2}$,

$$\frac{d^2z}{d\theta^2}+z\left\{1-\frac{af'(a)}{f(a)}\right\}=0.$$

If $\quad k=1-\frac{af'(a)}{f(a)}$, this becomes

$$\frac{d^2z}{d\theta^2}+kz=0. \tag{54}$$

When k is positive, the integral of this, by Art. 109, is

$$z=c\cos(\theta\sqrt{k}+\alpha),$$

or

$$u=a+c\cos(\theta\sqrt{k}+\alpha), \tag{55}$$

when c and α are arbitrary constants.

The greatest value of u is $a+c$; consequently, in order that the orbit should be approximately circular, it is necessary that c should be very small in comparison with a.

Again, supposing c positive, the greatest value of u has place when $\theta\sqrt{k}+\alpha=0$, and the least when $\theta\sqrt{k}+\alpha=\pi$; consequently the apsidal angle is

$$\frac{\pi}{\sqrt{k}} \text{ or } \frac{\pi}{\sqrt{1-a\frac{f'(a)}{f(a)}}}.$$

If k be negative, *i.e.* if $\frac{af'(a)}{f(a)}>1$, the integral of (54) is of the form

$$z=Ae^{\theta\sqrt{k}}+Be^{-\theta\sqrt{k}},$$

and therefore z would either increase or diminish indefinitely

with θ; and accordingly the orbit cannot be approximately circular in that case.

The value of k depends on the law of force: for example, if the force vary inversely as the n^{th} power of the distance, then

$$f(u) = \mu u^{n-2}, \quad \text{and} \quad \frac{af'(a)}{f(a)} = n - 2.$$

Accordingly, in this case, $k = 3 - n$.

Hence a *nearly circular orbit, having the centre of force in the centre, is impossible for laws of force which vary inversely as a higher power than the cube of the distance.*

When n is less than 3, the angle between the apsides is

$$\frac{\pi}{\sqrt{3-n}}.$$

For instance, if $n = 2$, the angle is π; this agrees with what has been already proved, as the orbit is a focal conic in this case.

Again, if $n = -1$, the angle is $\frac{1}{2}\pi$, as it ought to be, since the orbit is a central ellipse.

188. **Movable Orbits.**—If a central orbit be made to move in its own plane with an angular velocity proportional at each instant to that of the radius vector in the orbit, we can easily show—(1) that the new orbit is also a central orbit; (2) that the difference between the forces in the two orbits varies inversely as the cube of the distance from the centre of force. (Newton, *Principia*, lib. i., sect. 9.)

In a central orbit we have, in general,

$$r^2\frac{d\theta}{dt} = h, \text{ and } \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = P.$$

If now we make $\theta = k\theta'$, where k is constant, the former equation gives

$$r^2\frac{d\theta'}{dt} = \frac{h}{k} = h' \text{(suppose)}.$$

This shows that the point describes equal areas in equal times round the origin; accordingly the new path described by the point is also a central orbit.

Again, the second equation may be written

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta'}{dt}\right)^2 = P - (k^2 - 1)\, r\left(\frac{d\theta}{dt}\right)^2$$

$$= P + \frac{h^2 - h'^2}{r^3} = P' \text{ (suppose)};$$

hence $P' - P = \frac{h^2 - h'^2}{r^3}$: this shows that the difference between the forces in the fixed and movable orbits varies as $\frac{1}{r^3}$.

Hence, from any central orbit we can get another, called by Newton a *revolving orbit;* and the equation of the revolving orbit, in polar coordinates, is derived from that of the original by substituting $k\theta$ for θ; where the constant k is determined from the initial conditions.

For example, when $F = \frac{\mu}{r^2}$, we get a focal conic, whose equation is of the form

$$r = \frac{A}{1 + e\cos(\theta - a)};$$

hence, if $F = \frac{\mu}{r^2} + \frac{\mu'}{r^3}$, the equation of the orbit is of the form

$$r = \frac{A}{1 + e\cos(k\theta - a)}.$$

The apsidal angle in the new orbit is equal to that in the original orbit divided by k, as is readily seen. Newton applied this method to the investigation of the apsidal angle in the lunar orbit. His discussion is beyond the limits proposed in the present treatise. Moreover, the progression of the Moon's apse, thus determined by Newton, is but half its true amount.

Examples.

1. Find the law of force in a circle when the centre of force is situated on its circumference. *Ans.* $\frac{1}{r^5}$.

2. Investigate the motion of a body which is acted on by several centres of force varying directly as the distance; and show how to construct the position of the centre of the orbit.

3. In the same case, find the condition that the orbit should be a parabola.

4. Assuming that the law of force in a focal conic is that of the inverse square of the distance, show that the converse theorem can be immediately established, viz., that a particle attracted by a centre of force, varying according to that law, will describe a conic, having the centre of force in one of its foci.

5. A semi-ellipse is freely described by a particle under the action of a force parallel to its axis of figure; determine the requisite law of force, with the velocity of the particle on reaching or leaving either extremity of the semi-ellipse.

6. Prove that the law of force in an equiangular spiral is that of the inverse cube of the distance; and explain why we cannot assert, conversely, that a body acted on by such a force will describe an equiangular spiral.

7. If the velocity at each point in a central orbit be equal to that in the equidistant circle, prove that the orbit is an equiangular spiral for an attractive force.

By Art. 91 the velocity in the equidistant circle $= \sqrt{Fr}$. Again, by Art. 162, the velocity in the orbit $= \sqrt{F\gamma}$; therefore $r = \gamma = p\dfrac{dr}{dp}$. Hence $\dfrac{dr}{r} = \dfrac{dp}{p}$; therefore $r = kp$, and consequently the orbit is an equiangular spiral.

If the force be repulsive, the orbit is an equilateral hyperbola.

8. In general, if the velocity at each point in a central orbit be in a constant ratio to that in an equidistant circle; find the law of force and the equation of the orbit.

Let the constant ratio be represented by $1 : \sqrt{n}$; then, as in preceding example, we have

$$r = np\frac{dr}{dp}; \text{ hence } p = kr^n.$$

From this it is easily seen, as in Art. 162, that the equation of the orbit is of the form

$$r^{n-1} = a^{n-1} \cos (n - 1)\,\theta.$$

The law of force is readily found; for, in general,

$$F \propto \frac{1}{p^2\gamma} \propto \frac{1}{p^2 r} \propto \frac{1}{r^{2n+1}}.$$

9. In the same case, show that the velocity at each point in the orbit is that due to motion from an infinite distance, subject to the central force.

Here $v^2 = \dfrac{Fr}{n} = \dfrac{\mu}{nr^{2n}}$; hence, by (17) Art. 160, the velocity is that due to an infinite distance.

10. When the velocity and direction of motion at any point, as well as the centre and intensity of the force, are given, show how to find the radius of curvature of the orbit at the point.

11. A body is acted on by two attractive centres of force, of equal intensity; and also by a repulsive force from another centre, of double the intensity; the

forces varying directly as the distance. Prove that the orbit is a parabola, and show how to construct its focus and directrix when the initial velocity and direction of motion are given.

12. If a repulsive force vary as the inverse square of the distance, prove that the orbit is a branch of a hyperbola, having the centre of force in the focus external to the orbit.

13. A particle is acted on by a central repulsive force, which varies as the n^{th} power of the distance. If the velocity at any point be that due to motion from the centre of force; find the equation of the path.

Here, by (20), Art. 160, we have

$$v^2 = \frac{2\mu}{n+1} r^{n+1}, \text{ or } \frac{1}{p^2} = \frac{2\mu}{(n+1) h^2} r^{n+1};$$

or $$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{k^2}{u^{n+1}}, \text{ where } k^2 = \frac{2\mu}{h^2(n+1)};$$

therefore $$\frac{u^{\frac{n+1}{2}} du}{\sqrt{k^2 - u^{n+3}}} = d\theta;$$

hence we get $$u^{\frac{n+3}{2}} = a \cos \frac{n+3}{2} \theta.$$

14. If the velocity at each point in a central orbit varies directly as the distance from the centre of force, prove that the orbit is an equilateral hyperbola, and find the law of force.

15. Show that the velocity at any point in a focal parabola is to that in the equidistant circle as $\sqrt{2} : 1$. Tate and Steele Dynamics of part. 47-50

× 16. If the law of force be that of the inverse cube of distance, investigate the different varieties of orbit described.

Let $F = \mu u^3$, then equation (23) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} u; \text{ or } \frac{d^2u}{d\theta^2} + \left(1 - \frac{\mu}{h^2}\right) u = 0.$$

Accordingly, the equation of the orbit depends on the sign of $1 - \frac{\mu}{h^2}$, and therefore on the initial circumstances of the motion.

Suppose the particle projected initially at the distance R, with the velocity V, and in a direction which makes the angle ω with R; then

$$h = VR \sin \omega.$$

Again, if V' be the velocity in the equidistant circle, we have, Art. 89,

$$V'^2 = \frac{\mu}{R^2}; \quad \therefore \frac{\mu}{h^2} = \frac{\mu}{V^2 R^2 \sin^2 \omega} = \left(\frac{V'}{V \sin \omega}\right)^2.$$

Hence $1 - \frac{\mu}{h^2}$ is positive, zero, or negative, according as $V \sin \omega$ is greater, equal to, or less than V', the velocity in an equidistant circle.

(1) Let $V \sin \omega > V'$. In this case $1 - \frac{\mu}{h^2}$ is positive—equal k^2, suppose—and the equation may be written

$$\frac{d^2u}{d\theta^2} + k^2u = 0.$$

The integral of this is of the form

$$u = A \cos(k\theta + a).$$

A is plainly the maximum value of u; and therefore corresponds to an apsidal distance. Let a be this distance, and, if θ be measured from the apsidal line, the equation of the orbit is

$$r \cos k\theta = a. \qquad (1)$$

(2) Let $V \sin \omega = V'$, then $1 - \frac{\mu}{h^2} = 0$, and we have

$$\frac{d^2u}{d\theta^2} = 0;$$

this gives $u = A(\theta + a)$; and the equation of the orbit is reducible to

$$r\theta = \text{constant}, \qquad (2)$$

which represents the hyperbolic spiral.

(3) Let $V \sin \omega < V'$. If we multiply the equation

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} u$$

by $2du$, and integrate, we get

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{\mu}{h^2} u^2 + c,$$

where c is constant.

Hence $$v^2 = \mu u^2 + h^2 c.$$

Substituting the initial values, this gives

$$c = \frac{V^2 - V'^2}{h^2};$$

therefore $$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{\mu}{h^2} u^2 + \frac{V^2 - V'^2}{h^2}. \qquad (3)$$

The apse is determined by making $\frac{du}{d\theta} = 0$; consequently, since $\frac{\mu}{h^2} > 1$, the orbit has or has not an apse according as V is less or greater than V'.

Hence, if the *initial velocity be less than that in an equidistant circle*, the orbit is apsidal.

Suppose a to be the corresponding apsidal distance, then

$$\frac{V'^2 - V^2}{h^2} = \frac{1}{a^2}\left(\frac{\mu}{h^2} - 1\right),$$

and, making $\frac{\mu}{h^2} - 1 = k^2$, equation (3) becomes

$$\left(\frac{du}{d\theta}\right)^2 = \frac{k^2}{a^2}(a^2 u^2 - 1);$$

therefore $$\frac{du}{d\theta} = \frac{k}{a}\sqrt{a^2u^2 - 1}; \text{ or } \frac{a\,du}{\sqrt{a^2u^2 - 1}} = k\,d\theta.$$

The integral of this is

$$k\theta + \alpha = \log(au + \sqrt{a^2u^2 - 1}).$$

But if θ be measured from the apse, we have $\theta = 0$ when $au = 1$. Consequently $\alpha = 0$, and we have

$$au + \sqrt{a^2u^2 - 1} = e^{k\theta}.$$

Hence $$au = \tfrac{1}{2}(e^{k\theta} + e^{-k\theta}).$$

Here u increases with θ; and consequently the body, after leaving the apse, approaches nearer and nearer to the centre of force.

Secondly, if the initial velocity be equal to that in the equidistant circle, (3) becomes

$$\frac{du^2}{d\theta^2} = k^2u^2, \text{ or } \frac{du}{d\theta} = ku;$$

this gives $$u = ae^{k\theta},$$

the equiangular spiral (Ex. 7).

Thirdly, if V be greater than V', let

$$\frac{V^2 - V'^2}{h^2} = k^2\beta^2,$$

and equation (3) becomes

$$\left(\frac{du}{d\theta}\right)^2 = k^2(u^2 + \beta^2), \text{ or } \frac{du}{d\theta} = k\sqrt{u^2 + \beta^2}.$$

This, when integrated as above, gives

$$u + \sqrt{u^2 + \beta^2} = Ae^{k\theta},$$

and the curve is represented by the equation

$$2u = Ae^{k\theta} - \frac{\beta^2}{A}e^{-k\theta}.$$

The value of A can be readily determined from the initial conditions.

17. In elliptic motion about a centre of force in a focus, prove that $\int v\,ds$, taken through any arc, is proportional to the area subtended by the arc at the empty focus.

18. Prove that the expression for the central attraction for any law of force may be written in the form

$$F = \frac{h^2}{r^3} - \ddot{r}.$$

If we change the sign in the expression for the acceleration along the radius vector in Art. 28, we get

$$F = r\dot{\theta}^2 - \ddot{r}.$$

This assumes the proposed form on substituting for $\dot{\theta}$ its value $\frac{h}{r^2}$.

19. What would be the motion of a projectile if the force of gravity varied inversely as the cube of the height above a horizontal plane?

Here the path evidently lies in a vertical plane.

If the line of intersection of this plane with the horizontal plane be taken as the axis of x, and a vertical line as the axis of y, the equations of motion may be written

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = -\frac{\mu}{y^3};$$

therefore $$\frac{dx}{dt} = c, \quad \left(\frac{dy}{dt}\right)^2 = \frac{\mu}{y^2} + c',$$

where c and c' are constants which depend on the initial circumstances of the motion. Consequently

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{c^2}\frac{\mu + c'y^2}{y^2},$$

or $$\frac{ydy}{\sqrt{\mu + c'y^2}} = \frac{dx}{c}.$$

Hence we get $$\sqrt{\mu + c'y^2} = c'\frac{x}{c} + \text{const.}$$

Consequently the path is an ellipse or a hyperbola according as c' is negative or positive. The path is a parabola if $c' = 0$.

20. Prove by Newtonian methods that, if two bodies attract one another according to any law, they describe similar figures about their centre of inertia and about one another.

Neglecting the obliquity of the ecliptic, and the inclination and the eccentricity of the lunar orbit, show that, if we take the Sun's distance as 390 times that of the Moon, the Earth's mass as 79 times that of the Moon, and the lunar synodic period as 30 mean solar days; then the solar day is, to a near approximation, shorter at full Moon that at new Moon by one 468,000th part of a mean solar day. *Camb. Trip.*, 1882.

21. A material particle, moving freely in a plane, being supposed to describe a conic under the action of a central force emanating from any point in the plane; show that the force varies directly as the distance from the point, and inversely as the cube of its distance from the polar of the point with respect to the curve.

22. In free motion in a plane under the action of a central force varying according to any law, state and prove the effect on the trajectory (and on the motion in it) of an additional force emanating from the same centre, and varying inversely as the cube of the distance.

23. An ellipse of eccentricity e and a parabola have a common focus and latus rectum; and equal particles describe them under the action of forces, to the common focus, of the same absolute intensity. If the particles moving in the same direction meet at one extremity of the common latus rectum and coalesce, prove that their subsequent path will be an ellipse of eccentricity $\frac{1}{2}(1 \pm e)$, according as both foci of the ellipse do or do not lie within the parabola; and find its major axis. What will the path be, if the particles be moving in opposite directions when they meet? *Camb. Trip.*, 1879.

24. A body is revolving in an ellipse, whose eccentricity is $> \frac{1}{3}$, under the action of a force tending to the focus S; and when it is at a distance SP from S equal to the latus rectum, a blow is given to it perpendicular to SP, such that its new direction is perpendicular to the major axis. Show that the dimensions of the orbit are unaltered, but that the major axis is turned through an angle SPH, where H is the empty focus. *Id.*, 1882.

25. Find the laws of attraction for which the trajectories described round a centre of force are closed orbits. (Bertrand, *Comptes rendus*, 1873.)

If $2\mu \int \frac{\phi(u)\,du}{u^2} = F(u) + \text{const.}$, equation (23) gives

$$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{1}{h^2} F(u) + c, \tag{a}$$

where c is an arbitrary constant; therefore

$$\theta = \pm \int \frac{du}{\sqrt{\frac{1}{h^2} F(u) + c - u^2}}.$$

Again, let α, β represent the values of u which correspond to the apsidal distances, then α and β are roots of the equation

$$\frac{1}{h^2} F(u) + c - u^2 = 0.$$

Accordingly we must have

$$\frac{1}{h^2} F(\alpha) + c - \alpha^2 = 0, \quad \frac{1}{h^2} F(\beta) + c - \beta^2 = 0;$$

and if θ_0 be the apsidal angle, we get, abstraction being made of the sign,

$$\theta_0 = \int_\alpha^\beta \frac{du}{\sqrt{\frac{1}{h^2} F(u) + c - u^2}}.$$

Assuming $m\theta_0 = \pi$, then, for a closed orbit, m must be a commensurable number (Art. 184).

If $\frac{1}{h^2}$ and c be eliminated by aid of the two preceding equations, we obtain

$$\theta_0 = \int_\alpha^\beta \frac{du}{\sqrt{\beta^2 \frac{F(u) - F(\alpha)}{F(\beta) - F(\alpha)} - \alpha^2 \frac{F(u) - F(\beta)}{F(\beta) - F(\alpha)} - u^2}} = \frac{\pi}{m}; \qquad (c)$$

an equation which should hold for all values of α and β.

To determine the form of the function F, we suppose α and β very nearly equal, in which case the orbit is approximately circular.

Hence, from Art. 187, we get

$$m = \sqrt{k} = \sqrt{1 - \frac{\alpha F''(\alpha)}{F'(\alpha)}}, \text{ or } \frac{\alpha F''(\alpha)}{F'(\alpha)} = 1 - m^2.$$

Let $F'(\alpha) = V$, then $\frac{dV}{V} = \frac{F''(\alpha)}{F'(\alpha)} = (1 - m^2)\frac{d\alpha}{\alpha}$; hence

$$F'(\alpha) = V = C\alpha^{1-m^2},$$

where C is an arbitrary constant. (d)

From this we get $\qquad F(\alpha) = \frac{C}{2 - m^2}\alpha^{2-m^2} + \text{const}.$

We may assume the latter constant to be zero, since it disappears when we substitute in equation (c).

Again, since $\quad 2\mu \frac{\phi(u)}{u^2} = F'(u)$, we have $\phi(u) = C_1 u^{3-m^2}$,

where C_1 is arbitrary.

We next proceed to determine m from the condition that (c) must be satisfied for all values of α and β.

(1) Let $m^2 < 2$, and make $\alpha = 0$, and $\beta = 1$; then

$$F(\alpha) = 0, \quad F(\beta) = \frac{C}{2 - m^2}.$$

Substituting in (c), we obtain

$$\int_0^1 \frac{du}{\sqrt{u^{2-m^2} - u^2}} = \frac{\pi}{m}.$$

Again, if $u^{m^2} = z$,

$$\int_0^1 \frac{du}{u\sqrt{u^{-m^2} - 1}} = \frac{1}{m^2}\int_0^1 \frac{dz}{\sqrt{z(1-z)}} = \frac{\pi}{m^2}.$$

The condition gives $\frac{\pi}{m} = \frac{\pi}{m^2}$; therefore $m = 1$. Accordingly, the force varies as u^2, or as $\frac{1}{r^2}$.

(2) Let $m^2 > 2$; then, if $\alpha = 0$, we have $F(\alpha) = F(0) = -\infty$; and if $\beta = 1$ we have $F(\beta) = F(1) = \dfrac{C}{2 - m^2}$. Substitute in ($c$), and it becomes

$$\int_0^1 \frac{du}{\sqrt{1 - u^2}} = \frac{\pi}{m}; \quad \text{or } \frac{\pi}{2} = \frac{\pi}{m}; \quad \therefore\ m = 2,$$

in which case the force varies directly as the distance.

Hence, as M. Bertrand observes, "parmi les lois d'attraction qui supposent l'action nulle à une distance infinie, cella de la nature est la seule pour laquelle un mobile lancé *arbitrairement*, avec une vitesse inférieure à une certaine limite, et attiré vers un centre fixe, décrive nécessairement auteur de ce centre une courbe fermée. Toutes les lois d'attraction *permettent* des orbites fermées, mais la loi de la nature est la seule qui les *impose*."

26. Investigate the condition of stability of a circular orbit described about a centre of attraction in the centre of the circle.

Prove that if the attraction varies inversely as the fourth power of the distance, a particle describing a circle of radius a freely will be found ultimately describing either the curve

$$r = a\frac{\cosh\theta + 1}{\cosh\theta - 2}, \text{ or } r = a\frac{\cosh\theta - 1}{\cosh\theta + 2}.$$

CHAPTER VIII.

CONSTRAINED MOTION—MOTION IN A RESISTING MEDIUM.

SECTION I.—*Constrained Motion.*

189. **Motion on a Fixed Curve.**—When a particle is constrained to move, without friction, on a given fixed curve, the problem reduces to the determination of the velocity at any instant, as well as of the normal reaction of the curve. The motion may in this case be regarded as free by the introduction of the force of reaction of the curve, in addition to the external forces.

Hence, if N represents the normal reaction, the general equations of motion may be written, when referred to a rectangular system of axes,

$$m\frac{d^2x}{dt^2} = X + N\cos\alpha,\quad m\frac{d^2y}{dt^2} = Y + N\cos\beta,\quad m\frac{d^2z}{dt^2} = Z + N\cos\gamma, \tag{1}$$

where α, β, γ are the angles the normal reaction makes with the axes of coordinates; and X, Y, Z are the components of the external force, parallel to the axes of coordinates, respectively. If the first equation be multiplied by dx, the second by dy, and the third by dz, we get, on addition,

$$m\left(\frac{d^2x}{dt^2}dx + \frac{d^2y}{dt^2}dy + \frac{d^2z}{dt^2}dz\right) = X\,dx + Y\,dy + Z\,dz, \tag{2}$$

since $\cos\alpha\,dx + \cos\beta\,dy + \cos\gamma\,dz = 0$, as the direction of N is perpendicular to the tangent to the curve.

This gives on integration

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right\} = \int(X\,dx + Y\,dy + Z\,dz) + \text{const.} \tag{3}$$

Hence the velocity is given by the same equation as in the case of unconstrained motion (Art. 131).

For a conservative system of forces (Art. 125), the velocity v at any point can generally be found from this equation. For, let $X dx + Y dy + Z dz$ be the exact differential of the function $\phi(x, y, z)$; then if v' be the velocity at the point x', y', z', we have

$$\tfrac{1}{2} m (v^2 - v'^2) = \phi(x, y, z) - \phi(x', y', z'). \qquad (4)$$

Hence the velocity at any point is independent of the path described; and, accordingly, if different curves be drawn joining any two points, a particle starting from one of these points with a given velocity would arrive at the other point with the same velocity whatever path it described; friction being neglected.

Two of the preceding equations (1) are sufficient for a plane curve; for in this case N acts in the plane of the curve, and, by taking the axes of x and y in that plane, the third equation will disappear.

In the case of a central force, represented by $\mu\phi'(r)$, we have, as in Art. 131,

$$\tfrac{1}{2} m (v^2 - v'^2) = \mu(\phi(r) - \phi(r')).$$

Again, as in Art. 116, it is readily seen that the pressure on the curve in any case is the resultant of the centrifugal force and the normal component of the external forces.

The particle will leave the curve at the point for which the normal reaction becomes zero.

Examples.

1. A particle is constrained to move in a circle under the influence of a repulsive force, acting from a point on the circumference, and varying as the distance: find the pressure on the curve, the initial position being at the centre of force, and the particle starting from a state of rest.

Ans. $\dfrac{3\mu r^2}{2a}$, where r is the distance from the centre of force, and a the radius of the circle.

2. A particle is constrained to move in a logarithmic spiral, and is attracted to the pole of the spiral by a force varying inversely as the square of the distance. If the particle start from rest at the distance a from the pole, find the time of describing any portion of the curve.

Let μ denote the absolute force; then, by (5), we have

$$v^2 = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right),$$

or
$$\frac{ds}{dt} = \sqrt{2\mu}\sqrt{\frac{1}{r} - \frac{1}{a}}.$$

Again, if $r = ce^{\frac{\theta}{k}}$ be the equation of the spiral, we have

$$\frac{ds}{dt} = \frac{dr}{dt}\sqrt{1+k^2};$$

therefore
$$\frac{dr}{dt} = \sqrt{\frac{2\mu}{1+k^2}}\sqrt{\frac{1}{r} - \frac{1}{a}}.$$

Integrating, as in Art. 140, we get for the time of motion from the distance a to the distance r,

$$t = \sqrt{\frac{a(1+k^2)}{2\mu}}\left(a\cos^{-1}\sqrt{\frac{r}{a}} + \sqrt{r(a-r)}\right).$$

Also the whole time of motion to the centre is $\dfrac{\pi}{2}\sqrt{\dfrac{a^3(1+k^2)}{2\mu}}$.

It is readily seen that the problem of constrained motion in a logarithmic spiral, under the action of any central force directed to its pole, is reducible to free rectilinear motion under the action of a corresponding central force in the line of motion.

3. A particle under the action of gravity moves down the inner side of a smooth ellipse whose axis major is vertical. Being given its initial velocity, find where it will leave the ellipse.

Taking the centre as origin, and the axis major as axis of x, the value of x at the required point is given by the equation

$$2d = 3x - x^3\frac{e^2}{a^2},$$

where d is the height above the centre of the level line to which the velocity at each point is due.

4. In the same question find the least velocity at the lowest point of the ellipse in order that the particle should make a complete revolution in the curve.

Ans. $\sqrt{ga(5-e^2)}$.

190. Theorem of M. Ossian Bonnet.—If masses m, m', m'', &c., respectively subject to the action of forces, F, F', F'', &c., and starting all in the same direction from a point A, with velocities v_0, v_0', v_0'', &c., describe the same curve

ACB; then the same path will also be described by the mass M, when projected from the same point in the same direction, and subject to the action of all the forces, F, F', F'', &c., provided the initial *vis viva* MV_0^2 is equal to

$$mv_0^2 + m'v_0'^2 + m''v_0''^2 + \&c.,$$

the sum of the *vires vivæ* of the different masses. (Bonnet, *Liouville's Journal*, 1844.)

For, suppose the particle M constrained to move in the curve ACB, and let N be the normal reaction at any point; then, if the components of F, parallel to a rectangular system of axes, be respectively represented by X, Y, Z, those of F', by X', Y', Z', &c.; from (1), we have

$$M\frac{d^2x}{dt^2} = X + X' + X'' + \&c. + N\cos\alpha,$$

$$M\frac{d^2y}{dt^2} = Y + Y' + Y'' + \&c. + N\cos\beta,$$

$$M\frac{d^2z}{dt^2} = Z + Z' + Z'' + \&c. + N\cos\gamma,$$

and, as in (2), we have

$$d(MV^2) = 2dx\Sigma X + 2dy\Sigma Y + 2dz\Sigma Z.$$

But if v, v', v'', &c., be the velocities in the partial movements of m, m', m'', &c., at the same point,

$$d(mv^2) = 2(Xdx + Ydy + Zdz),$$

$$\&c., \qquad \&c., \qquad \&c.$$

Hence $\quad d(MV^2) = d(mv^2 + m'v'^2 + m''v''^2 + \&c.)$;

therefore $\qquad MV^2 = \Sigma(mv^2) + \text{constant},$

or $MV^2 = \Sigma mv^2$, from our hypothesis.

It is now easy to prove that the normal pressure N is zero at each point, and consequently that M would describe the curve ACB freely, under the combined action of all the forces.

For the force N is equal and opposite to the resultant of the centrifugal force, $\frac{MV^2}{\rho}$, and the several normal components of the forces, F, F', F'', &c.

$$\text{Again,} \qquad \frac{MV^2}{\rho} = \frac{mv^2}{\rho} + \frac{m'v'^2}{\rho} + \frac{m''v''^2}{\rho} + \text{\&c.}; \qquad (5)$$

but $\frac{mv^2}{\rho}$, $\frac{m'v'^2}{\rho}$, &c., are respectively equal and opposite to the normal components of F, F', F'', &c., because m, m', &c., describe the path ACB freely.

Hence there is equilibrium between the centrifugal force $\frac{MV^2}{\rho}$ and the total normal component of F, F', F'', &c.; and consequently $N = 0$.

In general, if the initial velocity of M do not satisfy the equation $MV_0^2 = \Sigma mv_0^2$, the normal pressure on the path ACB will vary directly as the curvature. For, from the preceding analysis,

$$N = \frac{MV^2 - \Sigma mv^2}{\rho} = \frac{MV_0^2 - \Sigma mv_0^2}{\rho}. \qquad (6)$$

Also, if one of the forces (F' suppose) be changed into its opposite, it is readily seen that the preceding theorem still holds, provided we change the sign of the corresponding term $(m'v'^2)$ in the expression $\Sigma(mv^2)$.

Examples.

1. A particle constrained to move in an ellipse is acted on by an attractive force directed to one focus, and a repulsive force from the other, whose intensities vary as the inverse square of the distance: if the absolute intensities of the forces be equal, find the pressure on the ellipse at any point during the motion.

2. Hence show that a particle placed at equal distances from two such centres of force will describe a semi-ellipse, under their joint action.

3. A particle moves under the attraction of two forces directed to the fixed points A and B, each varying according to the law of nature, and a third force, varying directly as the distance, directed to C, the middle point of AB; show that the particle can be projected from any point so as to describe an ellipse having A and B as its foci. Lagrange, *Méc. Anal.*, t. 2, § 83.

Ans. The initial velocity v_0 is given by the equation

$$v_0^2 = \frac{\mu f'}{af} + \frac{\mu' f}{af'} + \mu'' ff',$$

where μ, μ', μ'' denote the absolute forces for the centres A, B, C, respectively; f, f' the initial distances from A and B; and a the semiaxis major of the ellipse. The initial direction of motion must bisect the external angle formed by the lines joining A and B to the point of projection.

4. In the same case, if the particle be constrained to move in the ellipse, find the reaction R at any point during the motion.

$$\textit{Ans.}\quad R\rho = m\left(\frac{\mu f'}{af} + \frac{\mu' f}{af'} + \mu'' ff' - v_0^2\right),$$

where ρ is the radius of curvature at the point.

5. If a material particle, moving freely under the action of gravity, be disturbed by the action of a central force varying inversely as the square of the distance; determine the circumstances of its projection from a given point, in order that it may describe a parabola in a vertical plane having its focus at the centre of force.

191. **Motion on a Fixed Surface.**—If a particle be constrained to move on a smooth surface, the general equations of motion are plainly, as in (1),

$$m\frac{d^2x}{dt^2} = X + N\cos\alpha,\quad m\frac{d^2y}{dt^2} = Y + N\cos\beta,\quad m\frac{d^2z}{dt^2} = Z + N\cos\gamma,$$

where α, β, γ are the direction angles of the normal to the surface.

It is obvious that in this case also the velocity at any point is determined by the equation

$$\tfrac{1}{2}mv^2 = \int (X dx + Y dy + Z dz) + \text{const.} \qquad (7)$$

If gravity be the sole acting force, and the axis of z be taken in the vertical direction, our equations may be written

$$\frac{d^2x}{dt^2} = N\cos\alpha,\quad \frac{d^2y}{dt^2} = N\cos\beta,\quad \frac{d^2z}{dt^2} = N\cos\gamma - g. \qquad (8)$$

When the surface is one of revolution round a vertical axis, the normal at each point intersects that axis; and if n denote its length, we have

$$\cos\alpha = \frac{x}{n},\quad \cos\beta = \frac{y}{n}.$$

Hence the two former equations give

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0;$$

or, on integration,

$$x\frac{dy}{dt} - y\frac{dx}{dt} = c,$$

where c is a constant.

This equation shows that the point of projection on a horizontal plane describes equal areas in equal times round the point in which the axis of revolution meets the plane.

192. **Motion on a Spherical Surface.**—We shall apply what precedes to the motion of a particle under the action of gravity on a smooth sphere. This contains the general question of the motion of a simple pendulum, and is called the problem of the *spherical pendulum*. Taking the centre as origin, and the positive direction of the axis of z downwards, the equation of the sphere is

$$x^2 + y^2 + z^2 = a^2,$$

where a is the radius.

Also the general equations of motion may be written

$$\ddot{x} = N\frac{x}{a}, \quad \ddot{y} = N\frac{y}{a}, \quad \ddot{z} = N\frac{z}{a} + g,$$

adopting Newton's notation (Art. 23).

From the first two equations we get, as before,

$$x\dot{y} - y\dot{x} = c. \qquad (9)$$

Also, as in (7),

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = V_0^2 + 2g(z - a),$$

where V_0 represents the velocity corresponding to $z = a$.

Again, differentiating the equation of the sphere,

$$x\dot{x} + y\dot{y} + z\dot{z} = 0, \quad \text{or} \quad x\dot{x} + y\dot{y} = -z\dot{z}.$$

If this be squared and added to (9), when also squared, we get

$$(x^2 + y^2)(\dot{x}^2 + \dot{y}^2) = c^2 + z^2\dot{z}^2.$$

Hence $$(a^2 - z^2)\{V_0^2 + 2g(z - a) - \dot{z}^2\} = c^2 + z^2\dot{z}^2,$$

or $$a^2\dot{z}^2 = (a^2 - z^2)\{V_0^2 + 2g(z - a)\} - c^2. \qquad (10)$$

The subsequent investigation is simplified by supposing V_0 to correspond to the *lowest* point in the path of the particle; for, since the motion at that point is horizontal, we have $\dot{z} = 0$ when $z = \alpha$, and consequently

$$c^2 = (a^2 - \alpha^2)V_0^2 = 2gh(a^2 - \alpha^2),$$

if h be the height to which the velocity V_0 is due.

Substituting this value for c^2 in (10), we get

$$a^2\dot{z}^2 = 2g(\alpha - z)\{z^2 + h(z + \alpha) - a^2\}.$$

Again, the expression $z^2 + h(z + \alpha) - a^2$ may be written $(z - \beta)(z + \gamma)$, where

$$h = \frac{a^2 - \beta^2}{\alpha + \beta}, \text{ and } \gamma = \frac{a^2 + \alpha\beta}{\alpha + \beta}. \qquad (11)$$

Accordingly

$$a^2\dot{z}^2 = 2g(\alpha - z)(z - \beta)(z + \gamma);$$

therefore $$a\dot{z} = a\frac{dz}{dt} = -\sqrt{2g(\alpha - z)(z - \beta)(z + \gamma)}. \qquad (12)$$

The negative sign must be taken since z diminishes with t, which is reckoned from the instant the particle is in its lowest position.

Also, when $z = \beta$ we have $\dot{z} = 0$, and the motion is again horizontal. It is readily seen that during the motion z must lie between the limits α and β; and consequently the path of the particle is a tortuous curve lying between two horizontal lesser circles on the sphere; we accordingly may assume

$$z = \alpha\cos^2\phi + \beta\sin^2\phi, \qquad (13)$$

and, substituting in (12), get

$$2a\frac{d\phi}{dt} = \sqrt{2g(\alpha\cos^2\phi + \beta\sin^2\phi + \gamma)}.$$

Hence, since $t = 0$ when $\phi = 0$, we have

$$t = a\sqrt{\frac{2(a+\beta)}{g(a^2+2a\beta+\alpha^2)}}\int_0^\phi \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \qquad (14)$$

where $$k^2 = \frac{a-\beta}{a+\gamma} = \frac{a^2-\beta^2}{a^2+2a\beta+\alpha^2}.$$

Consequently the time of motion depends on an elliptic function, and is reducible to that of the description of a corresponding arc in a vertical circle (Arts. 101, 114).

Again, if T denote the *period of a vibration*, that is, the time of motion from a lowest to a consecutive lowest position, we have

$$T = 2a\sqrt{\frac{2(a+\beta)}{g(a^2+2a\beta+\alpha^2)}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}.$$

It may be observed that when $a = \beta$, we have $h = \dfrac{a^2-\alpha^2}{2a}$, and the question reduces to that of the conical pendulum, already considered in Art. 112.

Next let ψ be the angle that the vertical plane, passing through the centre and the position of the particle at any instant, makes with the plane of zx, then $y = x\tan\psi$; and consequently

$$c = x\frac{dy}{dt} - y\frac{dx}{dt} = x^2\frac{d}{dt}\left(\frac{y}{x}\right) = x^2\sec^2\psi\frac{d\psi}{dt}$$

$$= (x^2+y^2)\frac{d\psi}{dt} = (a^2-z^2)\frac{d\psi}{dt}. \qquad (15)$$

Also $$c = \sqrt{2gh(a^2-\alpha^2)} = \sqrt{\frac{2g(a^2-\alpha^2)(a^2-\beta^2)}{a+\beta}}; \qquad (16)$$

$$\therefore\ a\sqrt{\frac{(a^2-\alpha^2)(a^2-\beta^2)}{a+\beta}} = -(a^2-z^2)\sqrt{(a-z)(z-\beta)(z+\gamma)}\,\frac{d\psi}{dz},$$

and the angle ψ is represented by an elliptic function of the third species, thus

$$\psi = a\sqrt{\frac{(a^2-\alpha^2)(a^2-\beta^2)}{a+\beta}}\int_a^z \frac{-dz}{(a^2-z^2)\sqrt{(a-z)(z-\beta)(z+\gamma)}}. \qquad (17)$$

In the projection of the path on the horizontal plane through the centre, the greatest and least distances from the centre correspond to the greatest and least values of z, i. e. to $z = \alpha$ and $z = \beta$. These are called the apsidal distances, and the corresponding angle, the *apsidal angle* of the path. If Ψ be the apsidal angle its value is represented by the integral

$$\Psi = a\sqrt{\frac{(a^2-\alpha^2)(a^2-\beta^2)}{\alpha+\beta}}\int_\beta^\alpha \frac{dz}{(a^2-z^2)\sqrt{(\alpha-z)(z-\beta)(z+\gamma)}}. \quad (18)$$

193. **Small Oscillations.**—If the particle make a small oscillatory motion round the lowest point, we may, as a first approximation, make $\alpha = a$, $\beta = a$ in (14). This gives

$$k = 0, \text{ and } t = \phi\sqrt{\frac{a}{g}}. \quad (19)$$

Next, if $z = a\cos\theta$, $\alpha = a\cos\theta_0$, $\beta = a\cos\theta_1$, the equation

$$z = \alpha\cos^2\phi + \beta\sin^2\phi$$

gives $\theta^2 = \theta_0^2\cos^2\phi + \theta_1^2\sin^2\phi$, neglecting powers of θ, θ_0, θ_1 beyond the second.

Also (16) gives in this case

$$c = a^2\theta_0\theta_1\sqrt{\frac{g}{a}};$$

$\therefore$ by (15), we have

$$\frac{d\psi}{dt} = \frac{\theta_0\theta_1}{\theta^2}\sqrt{\frac{g}{a}} = \frac{\theta_0\theta_1}{\theta_0^2\cos^2\phi + \theta_1^2\sin^2\phi}\sqrt{\frac{g}{a}}.$$

Consequently, by (19),

$$\frac{d\psi}{d\phi} = \frac{\theta_0\theta_1}{\theta_0^2\cos^2\phi + \theta_1^2\sin^2\phi}.$$

Hence, by integration,

$$\tan\psi = \frac{\theta_1}{\theta_0}\tan\phi,$$

or
$$\frac{\sin\psi}{\cos\psi} = \frac{\theta_1 \sin\phi}{\theta_0 \cos\phi};$$
hence we have
$$\sin\psi = \frac{\theta_1 \sin\phi}{\theta},$$
and
$$\cos\psi = \frac{\theta_0 \cos\phi}{\theta}.$$

Moreover, to the same degree of approximation, we have
$$x = a\theta \cos\psi, \quad y = a\theta \sin\psi;$$
accordingly,
$$x = a\theta_0 \cos\phi, \quad y = a\theta_1 \sin\phi;$$
$$\therefore \quad \frac{x^2}{\theta_0^2} + \frac{y^2}{\theta_1^2} = a^2. \qquad (20)$$

This shows that the horizontal projection of the path is, approximately, an ellipse, whose semiaxes are $a\theta_0$ and $a\theta_1$.

The next approximation is given in the following examples.

The general problem of the spherical pendulum appears to have been first fully discussed by Lagrange: see *Méc. Anal.*, t. 2, sect. 8.

Examples.

1. If a particle perform small oscillations about the lowest point on a sphere, investigate its motion to an approximation of the second order.

It is here more convenient to transfer the origin to the lowest point on the sphere, and to take the positive direction of z *upwards*. Accordingly, we substitute $z = a - z'$, $\alpha = a - \alpha'$, $\beta = a - \beta'$, when equation (12) becomes
$$a\frac{dz'}{dt} = \sqrt{2g(z' - \alpha')(\beta' - z')\left(\frac{(2a - \alpha')(2a - \beta')}{2a - \alpha' - \beta'} - z'\right)}.$$

Hence, ***removing the accents***, we get

$$t = \tfrac{1}{2}\sqrt{\frac{a}{g}}\int_\alpha^z \frac{dz}{\sqrt{(z-\alpha)(\beta-z)\left(1 - \frac{z}{2a} + \frac{\alpha\beta}{2a(2a-\alpha-\beta)}\right)}},$$

where β and α represent the distances of the highest and lowest points in the path from the plane of xy.

Again, if α and β be both so small that their higher powers may be neglected, we obtain

$$t = \tfrac{1}{2}\sqrt{\frac{a}{g}}\int_\alpha^z \frac{dz}{\sqrt{(z-\alpha)(\beta-z)\left(1-\frac{z}{2a}\right)}}$$

$$= \tfrac{1}{2}\sqrt{\frac{a}{g}}\int_\alpha^z \frac{dz\left(1-\frac{z}{2a}\right)^{-\frac{1}{2}}}{(z-\alpha)^{\frac{1}{2}}(\beta-z)^{\frac{1}{2}}}$$

$$= \tfrac{1}{2}\sqrt{\frac{a}{g}}\int_\alpha^z \frac{dz}{\sqrt{(z-\alpha)(\beta-z)}} + \frac{1}{8\sqrt{ga}}\int_\alpha^z \frac{zdz}{\sqrt{(z-\alpha)(\beta-z)}},$$

neglecting the subsequent terms, since $\frac{z}{2a}$ is a very small fraction.

Hence, if $z = \alpha\cos^2\phi + \beta\sin^2\phi$, we get

$$t = \sqrt{\frac{a}{g}}\,\phi + \frac{1}{4\sqrt{ag}}\int_0^\phi (\alpha\cos^2\phi + \beta\sin^2\phi)\,d\phi$$

$$= \sqrt{\frac{a}{g}}\,\phi\left(1 + \frac{\alpha+\beta}{8a}\right) - \frac{\beta-\alpha}{8\sqrt{ag}}\sin\phi\cos\phi.$$

Consequently, if T be the whole time of motion from one lowest position to a consecutive one, we get

$$T = \pi\sqrt{\frac{a}{g}}\left(1 + \frac{\alpha+\beta}{8a}\right).$$

Again, to find the apsidal angle to the same degree of approximation. Transforming the origin in equation (16) to the lowest position, we readily obtain

$$\psi = a\sqrt{\alpha\beta}\int_\alpha^z \frac{dz}{z(2a-z)\sqrt{(z-\alpha)(\beta-z)\left(1 - \frac{z(2a-\alpha-\beta)}{(2a-\alpha)(2a-\beta)}\right)}}.$$

Hence, since as before we may take

$$\frac{2a - \alpha - \beta}{(2a - \alpha)(2a - \beta)} = \frac{1}{2a},$$

we get

$$\psi = a^{\frac{3}{2}} \sqrt{2\alpha\beta} \int_\alpha^z \frac{dz}{(2a - z)^{\frac{3}{2}} z \sqrt{(z - \alpha)(\beta - z)}}$$

$$= \tfrac{1}{2} \sqrt{\alpha\beta} \int_\alpha^z \frac{dz \left(1 - \frac{z}{2a}\right)^{-\frac{3}{2}}}{z \sqrt{(z - \alpha)(\beta - z)}}$$

$$= \tfrac{1}{2} \sqrt{\alpha\beta} \int_\alpha^z \frac{dz}{z \sqrt{(z - \alpha)(\beta - z)}} + \frac{3}{8} \frac{\sqrt{\alpha\beta}}{a} \int_\alpha^z \frac{dz}{\sqrt{(z - \alpha)(\beta - z)}},$$

neglecting the subsequent terms as before.

Substituting $\alpha \cos^2\phi + \beta \sin^2\phi$ for z, we obtain

$$\psi = \tan^{-1}\left(\sqrt{\frac{\beta}{\alpha}} \tan \phi\right) + \frac{3}{4} \frac{\sqrt{\alpha\beta}}{a} \phi.$$

Hence, taking ϕ between the limits 0 and $\frac{\pi}{2}$, the apsidal angle is given, approximately, by the equation

$$\Psi = \frac{\pi}{2}\left(1 + \frac{3}{4} \frac{\sqrt{\alpha\beta}}{a}\right).$$

This shows that in the approximate elliptic path the apse continually progresses.

Again, if p, q denote the small apsidal distances, or the semidiameters of the approximate elliptic path, we get

$$\Psi = \frac{\pi}{2}\left(1 + \frac{3}{8} \frac{pq}{a^2}\right).$$

Accordingly, the rate of progression of the apse varies approximately as the area of the projection of the path.

2. Prove that the pressure on the sphere is given by the equation

$$N = \frac{mg}{a}\left\{3z - \frac{2}{\alpha + \beta}(\alpha^2 + \alpha\beta + \beta^2 - a^2)\right\}.$$

3. If a particle be projected with a given velocity along the horizontal great circle of a smooth hollow sphere, find at what point its vertical velocity will be greatest.

Ans. $z = \frac{\sqrt{h^2 + 3a^2} - h}{3}$, h being the height due to the velocity of projection.

4. A particle is projected horizontally along the interior surface of a fixed smooth hemisphere, the axis of which is vertical, and vertex downwards. Given the point of projection, determine the velocity so that the particle may ascend exactly to the rim of the hemisphere.

Ans. $a\sqrt{\frac{2g}{a}}$.

5. If a particle move on the interior surface of a paraboloid of revolution, whose axis is vertical, prove that the velocity at the highest point in the path is that due to the height of the lowest point above the vertex of the paraboloid; and similarly for the velocity at the lowest point.

6. In the last question show that the pressure at any point P varies as the curvature of the meridian at that point; and that the resolved vertical pressure is to the weight of the particle as $SL \times SM : SP^2$, where L and M are the highest and lowest points of the path, and S the focus.

Section II.—*Rectilinear Motion in a Resisting Medium.*

194. If a mass m be supposed to move in a straight line, without rotation, in a resisting medium, the resistance is a function of the velocity of the body. If the resistance be represented by $\phi(v)$, the equation of motion becomes

$$m\frac{dv}{dt} = F - \phi(v),$$

where F is the external force acting along the right line.

It is usual to assume, with Newton, that $\phi(v) = \mu v^2$, where μ is a constant depending on the density of the medium and on the area (S) of the greatest section of the body taken perpendicular to the direction of motion.

Hence we get

$$m\frac{dv}{dt} = F - \mu v^2.$$

If we suppose F constant, and make

$$F = \mu V^2, \tag{1}$$

we get

$$m\frac{dv}{dt} = \mu(V^2 - v^2). \tag{2}$$

If the initial velocity be less than V, it is obvious that the velocity increases so long as it is less than V: this gives V

as the limit to which the velocity approaches. For this reason V is called the *terminal velocity* of the body.

Also, since $$\frac{1}{V^2-v^2} = \frac{1}{2V}\left\{\frac{1}{V+v} + \frac{1}{V-v}\right\},$$

the preceding equation gives

$$t = \frac{mV}{2F}\log\left(\frac{V+v}{V-v}\right). \qquad (3)$$

No constant is added since we suppose t reckoned from the position of rest.

Equation (3) shows that, while v increases with t, yet when $v = V$ we should have $t = \infty$. Accordingly the body requires an infinite time before arriving at its terminal velocity.

195. **Vertical Motion.**—One of the most important cases is that of a body falling vertically in a resisting medium. In this case $F = mg$, and equation (3) becomes

$$t = \frac{V}{2g}\log\left(\frac{V+v}{V-v}\right). \qquad (4)$$

This gives $$\frac{V+v}{V-v} = e^{\frac{2gt}{V}}.$$

Hence $$v = V\frac{e^{\frac{gt}{V}} - e^{-\frac{gt}{V}}}{e^{\frac{gt}{V}} + e^{-\frac{gt}{V}}} = V\tanh\frac{gt}{V}. \qquad (5)$$

Again, since $$v = \frac{dx}{dt},$$

we get $$x = \frac{V^2}{g}\log\left(\frac{e^{\frac{gt}{V}} + e^{-\frac{gt}{V}}}{2}\right),$$

when x is measured from the position of rest.

This may be written in the form

$$x = \frac{V^2}{g} \log \cosh \frac{gt}{V} \tag{6}$$

Again we may write $\mu = AS$, where A is a constant depending on the density of the medium.

Hence from (1) we get

$$V = \sqrt{\frac{W}{\mu}} = \sqrt{\frac{W}{AS}}, \tag{7}$$

where W denotes the weight of the body.

This shows that, W remaining the same, the value of V can be increased by diminishing the area of the transverse section.

In the case of a homogeneous sphere of radius r, we have $W = \frac{4}{3}\pi r^3 p$, where p is the weight of a unit of volume; also $S = \pi r^2$; therefore

$$V = \sqrt{\frac{4pr}{3A}}.$$

Hence we see that for spheres of the same density that of the greater radius has the greater terminal velocity, and we can readily compare the vertical motions of different spheres in the same resisting medium.

Next, for a body projected vertically upwards in a resisting medium the equation of motion is

$$\frac{dv}{dt} = -g\left(1 + \frac{v^2}{V^2}\right),$$

whence
$$dt = -\frac{V^2}{g}\frac{dv}{v^2 + V^2}.$$

Accordingly, if V_0 be the initial velocity, we find

$$t = \frac{V}{g}\left(\tan^{-1}\frac{V_0}{V} - \tan^{-1}\frac{v}{V}\right).$$

From this equation the velocity at any instant can be determined.

Also, since $v = 0$ at the highest point, the time of ascent to that point is represented by $\frac{V}{g}\tan^{-1}\frac{V_0}{V}$.

Again
$$dx = -\frac{V^2}{g}\frac{v\,dv}{v^2 + V^2}.$$

Hence, if x be measured upwards from the point of projection, we have
$$x = \frac{V^2}{2g}\log\frac{V_0^2 + V^2}{v^2 + V^2}.$$

If h be the height of ascent, we get
$$h = \frac{V^2}{2g}\log\left(\frac{V_0^2 + V^2}{V^2}\right). \qquad (8)$$

If the time t be reckoned from the instant at which the body is at its highest point, we have
$$v = V\tan\frac{gt}{V}. \qquad (9)$$

The downward motion is given by the former investigation.

EXAMPLES.

1. Find a vertical curve such that the time of describing any arc, measured from a fixed point, shall be equal to that of describing the chord of the arc.

Taking the origin at the fixed point, the time down a chord r, whose inclination to the vertical is θ, as in Art. 46, is

$$\sqrt{\frac{2r}{g\cos\theta}}$$

Also the time of descending the arc is

$$\frac{1}{\sqrt{2g}}\int_{\theta_0}^{\theta}\frac{\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}}{\sqrt{r\cos\theta}}\,d\theta,$$

where θ_0 is the value of θ when $r = 0$.

Hence, since the times are the same for all chords, we get, by differentiation,

$$\frac{r\sin\theta+\cos\theta\frac{dr}{d\theta}}{\cos\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}.$$

This gives $$\frac{1}{r}\frac{dr}{d\theta}=\cot 2\theta;$$

hence we get $$r^2 = a^2\sin 2\theta,$$

where a is a constant. Accordingly the curve is a Lemniscate.

2. Investigate the corresponding problem when the acting force is proportional to the distance from a fixed point.

Let A be the position of the fixed point, O the point of departure of the particle, P its position at any instant, $\theta = \angle POA$, $OA = a$; then we find, without difficulty, that the time t_1, of describing OP, when the absolute force is taken as unity, is given by

$$t_1 = \sin^{-1}\frac{r-a\cos\theta}{a\cos\theta}+\frac{\pi}{2}.$$

Also the time of describing the arc OP is

$$t_2=\int\frac{\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}}{\sqrt{2ar\cos\theta-r^2}}\,d\theta.$$

Hence, since $t_1 = t_2$, we have

$$\sqrt{2ar\cos\theta-r^2}\,\frac{d}{d\theta}\left(\sin^{-1}\frac{r-a\cos\theta}{a\cos\theta}\right)=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2};$$

therefore $$\frac{dr}{d\theta} + r\tan\theta = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2};$$

from which we get $r^2 = a^2 \sin 2\theta$. This represents a lemniscate also, as in the previous question.

3. If the motion of a conical pendulum be slightly disturbed, prove that the period of a vibration is $\frac{2\pi a}{\sqrt{a^2+3b^2}}\sqrt{\frac{b}{g}}$, and the corresponding apsidal angle $\pi\frac{a}{\sqrt{a^2+3b^2}}$, where b is the distance from the centre to the plane of the conical pendulum.

4. A particle is projected from a given point in a horizontal direction along the surface of a smooth sphere; find the velocity of projection in order that the particle should rise to a given height on the surface before commencing to descend.

5. A particle is constrained to move in a smooth circle, under the action of a central force which varies directly as the distance. If the time of describing any arc be constant, prove that its chord envelops a circle.

Townsend, *Educ. Times*, 1875.

6. If a particle describe a curve freely under the combined action of the forces F, F', &c., where F, F', &c., act along r, r', &c., prove that the equation

$$\gamma\phi d\left(\frac{F}{\phi}\right) + \gamma'\phi' d\left(\frac{F'}{\phi'}\right) + \&c. = 0$$

must be satisfied at every point of the curve, where ϕ, ϕ', &c., denote the forces respectively co-directional with F, F', &c., under which *singly* the given curve would be described; and γ, γ', &c., are the corresponding semichords of the circle of curvature at the point.

Curtis, *Messenger of Mathematics*, 1880.

Here, it is easily seen by equation (25), Art. 162, that

$$v^2 = F\gamma + F'\gamma' + \&c.$$

Also, by (13), Art. 160, $$v\,dv = -F dr - F' dr' - \&c.$$

Hence $$\Sigma(F d\gamma + \gamma dF) + 2\Sigma F dr = 0,$$

or $$\Sigma\{F(d\gamma + 2dr) + \gamma dF\} = 0.$$

Hence, in particular, we have

$$\phi(d\gamma + 2dr) + \gamma d\phi = 0, \&c.$$

or $$\frac{d\gamma + 2dr}{\gamma} = \frac{-d\phi}{\phi}, \&c.;$$

$$\therefore \quad \Sigma\gamma\left(dF - \frac{F d\phi}{\phi}\right) = 0, \quad \text{or} \quad \Sigma\gamma\phi d\left(\frac{F}{\phi}\right) = 0.$$

This theorem plainly contains as a particular case that given in Art. 190.

7. Apply the preceding to the case of a conic described under the action of forces, F, F', directed to its foci.

Here $$\phi = \frac{\mu}{r^2}, \quad \phi' = \frac{\mu'}{r'^2}, \quad \gamma = \gamma';$$

therefore $$\frac{1}{r^2}\frac{d}{dr}(Fr^2)\,dr + \frac{1}{r'^2}\frac{d}{dr'}(F'r'^2)\,dr' = 0,$$

or, since $dr + dr' = 0$,

$$\frac{1}{r^2}\frac{d}{dr}(Fr^2) = \frac{1}{r'^2}\frac{d}{dr'}(F'r'^2).$$

This is satisfied by the equations

$$\frac{1}{r^2}\frac{d}{dr}(Fr^2) = f_1(r) + f_2(2a - r),$$

$$\frac{1}{r'^2}\frac{d}{dr'}(F'r'^2) = f_2(r') + f_1(2a - r'),$$

where f_1 and f_2 are both arbitrary functions.

If we assign the same form (f) to f_1 and f_2, we obtain as a particular solution

$$\frac{1}{r^2}\frac{d}{dr}(Fr^2) = f(r) + f(2a - r),$$

$$F = \frac{1}{r^2}\int r^2\{f(r) + f(2a - r)\}\,dr, \text{ \&c.}$$

If any particular form be assigned to f, a corresponding form of F, as also of F', will result.

8. As an example of the preceding, show that a particle can be made to describe an ellipse freely under the action of forces,

$$\lambda r + \frac{\mu}{r^2}, \quad \lambda r' + \frac{\mu'}{r'^2},$$

directed to its foci.

The student is referred to Professor Curtis' Paper for additional applications.

9. A spherical particle moves within a smooth rectilinear tube, which revolves about one extremity with a uniform angular velocity in a horizontal plane; find the motion of the particle.

Let ω be the angular velocity of the tube, and r the distance of the particle, at any time t, from the fixed extremity of the tube; then, since the force acting on the particle is always perpendicular to r, we have (Art. 28),

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = 0, \quad \text{or} \quad \frac{d^2r}{dt^2} - \omega^2 r = 0.$$

Hence $r = ce^{\omega t} + c'e^{-\omega t}$. If $r = a$, and $\frac{dr}{dt} = b$, when $t = 0$, we get

$$2\omega r = (a\omega + b)\, e^{\omega t} + (a\omega - b)\, e^{-\omega t}.$$

10. Consider the same problem if the tube be supposed to revolve uniformly in a vertical plane.

Here, if the time be reckoned from the instant that the tube was horizontal, the equation of motion is

$$\frac{d^2 r}{dt^2} - \omega^2 r = -g \sin \omega t.$$

The integral of this is

$$r = Ce^{\omega t} + C'e^{-\omega t} + \frac{g}{2\omega^2} \sin \omega t,$$

and the constants can be determined from the initial conditions.

11. Two spheres of the same diameter, but of different weights, fall freely in air; find the ratio of the maximum velocities they will attain, stating clearly what assumptions you make. *Lond. Univ.*, 1881.

12. Explain what is meant by the terminal velocity of a body in a resisting medium.

If the resistance vary as the square of the velocity and the body move in a vertical line, prove that at the time t, reckoned from the instant at which the body is at its highest position, its depth x below this position is given by

$$x = \frac{\omega^2}{g} \log \sec \frac{gt}{\omega},$$

when ascending, and by

$$x = \frac{\omega^2}{g} \log \cosh \frac{gt}{\omega},$$

when descending; ω denoting the terminal velocity in the medium.
Lond. Univ., 1883.

13. If a body be projected vertically upwards in a resisting medium with its terminal velocity for the medium, determine the height of its ascent, and the time of reaching the highest point.

Prove that, if an engine can pull a train of W tons at a velocity V on the level, against resistances varying as the square of the velocity, the engine exerting a constant pull of P tons: then up an incline α to the horizon the maximum velocity will fall to $V\sqrt{(1 - W \sin\alpha / P)}$, and that down the incline without steam the terminal velocity is $V\sqrt{(W \sin\alpha / P)}$.

Prove that, if on a long railway journey, performed with average velocity V, the actual velocity v varies from its mean value by a periodic function of the time, say $v = V + U \sin nt$, the average horse-power and consumption of fuel is to that required to take the train with uniform velocity V as

$$1 + \tfrac{3}{2}\, U^2 / V^2 : 1.$$

Lond. Univ., 1887.

CHAPTER IX.

THE GENERAL DYNAMICAL PRINCIPLES.

196. **D'Alembert's Principle.**—If a system of material points connected together in any way, and subject to any constraints, be in motion under the influence of any forces, each point of the system has at any instant a certain acceleration. If now to each point an acceleration were applied equal and opposite to its actual acceleration, the velocities of all the points of the system would become constant—in other words, each point would move as if free and unacted on by any force whatever; that is, the applied accelerations, the external forces, and the constraints and mutual or internal forces of the system, would equilibrate each other.

Stated in algebraical language, the principle which is given above may be enunciated as follows:—If the coordinates of any particle m of a material system be x, y, z, and the external forces there applied X, Y, Z; the system of forces,

$$X_1 - m_1\frac{d^2x_1}{dt^2},\quad Y_1 - m_1\frac{d^2y_1}{dt^2},\quad Z_1 - m_1\frac{d^2z_1}{dt^2},$$

$$X_2 - m_2\frac{d^2x_2}{dt^2},\quad Y_2 - m_2\frac{d^2y_2}{dt^2},\quad Z_2 - m_2\frac{d^2z_2}{dt^2},\ \text{\&c.},$$

acting at the points $x_1y_1z_1$, $x_2y_2z_2$, &c., will be in equilibrium, in virtue of the constraints and mutual reactions of the system.

The force whose components are $-m\frac{d^2x}{dt^2}$, $-m\frac{d^2y}{dt^2}$, $-m\frac{d^2z}{dt^2}$ is called the force of inertia of the mass m, and D'Alembert's Principle (as stated in Article 71) simply expresses that—

The applied forces and the forces of inertia in any system are in equilibrium.

In applying D'Alembert's Principle, we may, as in Statics, consider the constraints of the system either as geometrical conditions, or else substitute for them unknown forces. In the algebraical statement just given, the former plan has been adopted; but if we choose to adopt the latter, we have merely to make X, Y, Z, &c., include not only the applied forces, but also the stresses arising from the constraints.

If the Statical Principle of Virtual Velocities be employed, we have for D'Alembert's Principle the concise mode of expression given by Lagrange in his *Mécanique Analytique*, viz.:—

$$\Sigma \left\{\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\frac{d^2z}{dt^2}\right)\delta z\right\} = 0. \quad (1)$$

This equation may also be written

$$\Sigma m(\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z) = \Sigma(X\delta x + Y\delta y + Z\delta z), \quad (2)$$

a form which is often more convenient than (1).

If the forces X, Y, Z, &c., constitute a conservative system, Art. 124, we may write

$$\Sigma(X\delta x + Y\delta y + Z\delta z) = \delta\Upsilon,$$

and (2) becomes in this case

$$\Sigma m(\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z) = \delta\Upsilon. \quad (3)$$

197. **D'Alembert's Principle for Impulses.**—As has been stated already in Article 66, an Impulsive or Instantaneous Force is a force which produces a finite change of velocity in a time so short that in it no sensible change of velocity is produced by the action of the forces which are not impulsive. If the constraints and connections of a system be regarded as giving rise to forces, these forces may be impulsive or not, according to the nature of the constraint. For example, a blow given to a body which is resting on an immovable surface produces an impulsive reaction, provided the blow is not tangential to the surface; but a sudden jerk to a body attached to the end of an extensible elastic string produces no impulsive reaction. It is important to observe that

each point of the system may be regarded as occupying the same position in space at the end as at the beginning of the time during which the impulsive forces have acted. In other words, the velocities of the various points may change by a finite amount, but the positions can only change by an infinitely small amount during the time under consideration.

If u', v', w' be the components of the velocity of any point, whose coordinates are x, y, z, before the action of the impulsive forces; and u, v, w the corresponding velocities after their action; and $\underset{\cdot}{X}$, $\underset{\cdot}{Y}$, $\underset{\cdot}{Z}$ be the components of the impulse which has acted at this point, D'Alembert's Principle as applied to impulsive forces may be expressed in the form—

$$\Sigma m\{(u-u')\delta x + (v-v')\delta y + (w-w')\delta z\} = \Sigma(\underset{\cdot}{X}\delta x + \underset{\cdot}{Y}dy + \underset{\cdot}{Z}\delta z). \tag{4}$$

The truth of the Principle in the present case can be established by reasoning similar to that employed in the preceding Article.

It may also be derived from the Principle applied to continuous forces, by considering the impulsive forces as continuous forces of great magnitude acting for a very short time. In fact, if we multiply the equation

$$\Sigma\left\{\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\frac{d^2z}{dt^2}\right)dz\right\} = 0$$

by dt, and integrate between the limits t and t'; if the interval $t - t'$ be sufficiently short, the system has not sensibly altered its position, and therefore δx, &c., are the same at the end of the time as at the beginning, and we have

$$\Sigma\left\{\left[\int_{t'}^{t} Xdt - m\left(\frac{dx}{dt} - \left(\frac{dx}{dt}\right)'\right)\right]\delta x + \text{\&c.}\right\} = 0.$$

Now, if X be the component of a continuous force, $\int_{t'}^{t} Xdt$ is insensible; and if X be the component of an impulsive force, $\int_{t'}^{t} Xdt$ is the component of the impulse along the axis of x,

which may be denoted by $\underset{.}{X}$; hence, as

$$\frac{dx}{dt} = u, \quad \left(\frac{dx}{dt}\right)' = u', \text{ \&c.,}$$

we immediately obtain equation (4).

198. **Initial Motion.**—If a system start from rest under the action of given impulses, equation (4), Art. 197, becomes

$$\Sigma m\,(u\delta x + v\delta y + w\delta z) = \Sigma\,(\underset{.}{X}\delta x + \underset{.}{Y}\delta y + \underset{.}{Z}\delta z), \qquad (5)$$

where u, v, w are the components of the initial velocity of the point xyz. Now as δx, δy, δz are any arbitrary displacements of this point, consistent with the conditions of the system, we may, if the equations of condition do not involve the time explicitly, substitute for δx, δy, δz the actual displacements of the point (see Art. 200). Hence, as actual displacements when divided by the element of time become velocities, we may substitute for δx, δy, δz the components u', v', w', of the velocity of xyz in *any actual* motion of the system. Thus we obtain

$$\Sigma m\,(uu' + vv' + ww') = \Sigma\,(\underset{.}{X}u' + \underset{.}{Y}v' + \underset{.}{Z}w'). \qquad (6)$$

Examples.

1. If the same system be set in motion successively by two different impulses applied at the same point, each impulse is proportional to the velocity in the direction of the other which it imparts to its point of application.

Let these velocities be $\dot{q}$ and $\dot{p}'$, and let $\underset{.}{X}$, $\underset{.}{Y}$, $\underset{.}{Z}$; $\underset{.}{X}'$, $\underset{.}{Y}'$, $\underset{.}{Z}'$ be the components of the impulses P and Q, and u, v, w; u', v', w' the components of the initial velocities of the point of application, then,

$$\underset{.}{X}u' + \underset{.}{Y}v' + \underset{.}{Z}w' = \Sigma m(uu' + vv' + ww') = X'u + Y'v + Z'w;$$

but $\quad P\dot{p}' = \underset{.}{X}u' + \underset{.}{Y}v' + \underset{.}{Z}w', \quad$ and $\quad Q\dot{q} = X'u + Y'v + Z'w,$

whence $\quad P : Q :: \dot{q} : \dot{p}'.$

2. In any system at rest, if we suppose an impulse P applied at a point A, and an impulse P' applied at a point B; prove that

$$P : P' = v : v',$$

where v is the component, in the direction of P', of the velocity of the point B due to the impulse P; and v' is the similar component of velocity of the point A.

199. **Energy of Initial Motion.**—If T be the initial kinetic energy of a system set in motion by given impulses, by substituting u, v, w for $\delta x, \delta y, \delta z$ (in 5) we obtain

$$2T = \Sigma m(u^2 + v^2 + w^2) = \Sigma(\underset{\cdot}{X}u + \underset{\cdot}{Y}v + \underset{\cdot}{Z}w). \qquad (7)$$

*Bertrand's Theorem.**—If a system start from rest under the action of given impulses, every additional constraint diminishes the initial kinetic energy.

Let u', v', w' be the initial velocities of the point xyz under the action of the given impulses when the additional constraints are imposed; and u, v, w the initial velocities when the system is *free* from these constraints, then, $u'dt, v'dt, w'dt$ are possible displacements in the unconstrained as well as in the constrained system. Hence, substituting u', v', w' for $\delta x, \delta y, \delta z$ in equation (5) we obtain

$$\Sigma m(uu' + vv' + ww') = \Sigma(\underset{\cdot}{X}u' + \underset{\cdot}{Y}v' + \underset{\cdot}{Z}w').$$

But, by (7), $\Sigma m(u'^2 + v'^2 + w'^2) = \Sigma(\underset{\cdot}{X}u' + \underset{\cdot}{Y}v' + \underset{\cdot}{Z}w')$;
thus we have

$$\Sigma m\{(u-u')^2 + (v-v')^2 + (w-w')^2\} = \Sigma m(u^2 + v^2 + w^2)$$
$$- 2\Sigma m(uu' + vv' + ww') + \Sigma m(u'^2 + v'^2 + w'^2)$$
$$= 2T - 4T' + 2T' = 2T - 2T'. \qquad (8)$$

Hence, we see that the energy of the unconstrained exceeds that of the constrained motion by the energy of the motion which must be combined with either to produce the other.

Thomson's Theorem.†—If impulses are applied only at points where the velocities are prescribed, additional constraints increase the initial kinetic energy.

Here, when additional constraints are imposed, the impulses are supposed to be altered in such a manner as still to produce the prescribed velocities in the assigned points; then, u', v', w' being, as before, the velocities belonging to the *constrained* motion, we have, since in the present case the

* *Liouville*, tome septième (1842), p. 165.
† *Proceedings* of Royal Society of Edinburgh, April, 1863.

The equation of *vis viva* has been already obtained for a rigid body in a different manner in Article 132.

The equation of *vis viva* is one of the most important in Dynamics, and is to a great extent the foundation of the Theory of Energy. It will be more fully considered in a future chapter.

201. **Of the Forces which enter the Equation of Vis Viva.**—From the mode in which the equation of *vis viva* has been deduced from D'Alembert's Principle, it is plain that in the case of a rigid body the right-hand side contains only the applied forces, and that reactions by which geometrical conditions may be replaced do not enter therein. The reactions of fixed points, fixed surfaces, &c., are thus excluded: and further, if during the motion the direction of a force be at each instant at right angles to the line in which its point of application is moving, such a force does not enter the equation of *vis viva* (Art. 122).

When two surfaces roll on one another without slipping, the *relative* tangential displacement of the two points in contact is zero. Now, if F be the tangential force of friction developed between them, the element of work done by F on one body is Fdf_1, and that done on the other is $-Fdf_2$, df_1 and df_2 being the projections on the direction of F of the small motions of the two points in contact. Hence the whole work done by the tangential friction is $F(df_1 - df_2)$; but $df_1 - df_2$ is the relative tangential displacement of the points of the surfaces which are in contact. Hence the whole work done by the tangential force of friction in *pure rolling* is zero, or this force in the case supposed has no effect on the equation of *vis viva*. If one of the surfaces be fixed, a similar result obviously holds good.

When two surfaces are in permanent contact, the normal reaction between them never enters the equation of *vis viva*. For, if the motion be pure slipping, the relative velocity of the points of application of the mutual reaction is altogether tangential. If the motion be either rolling and slipping, or pure rolling, the relative normal velocity of the points in contact is still zero, or at least infinitely small, and the relative normal displacement is an infinitely small quantity of the second order; that is, $dr_1 - dr_2 = 0$, where dr_1 and dr_2 are

the projections on the common normal of the small motions of the points in contact. Hence $R(dr_1 - dr_2)$, which is the whole work done by the mutual normal reaction, is equal to zero.

The work done in the element of time by a mutual force between two bodies is always of the form $R(dr_1 - dr_2)$, where $\frac{dr_1}{dt}$ and $\frac{dr_2}{dt}$ are the velocities, in the direction of the joining line, of the points between which the force R acts. If r be the distance between these points, the work done by the mutual action is therefore $R\,dr$.

If R tends to increase the relative velocity in its own direction which already exists, $R\,dr$ is positive. If on the other hand R tends to diminish this velocity, $R\,dr$ is negative. This readily appears from the following considerations:—

If the mutual action tends to increase the velocity $\frac{dr_1}{dt}$, it tends to diminish $\frac{dr_2}{dt}$, and therefore the element of work done by it is $R(dr_1 - dr_2)$; but this is positive if $\frac{dr_1}{dt} > \frac{dr_2}{dt}$, and negative if $\frac{dr_2}{dt} > \frac{dr_1}{dt}$. In the first case the mutual action tends to increase the relative velocity in its own direction, and in the second case to diminish this velocity. Also, if the mutual action tends to diminish $\frac{dr_1}{dt}$ similar reasoning applies.

202. **Effect of Impulses on Vis Viva.**—The change of *vis viva* resulting from impulses may be investigated by means of equation (4), Art. 197.

In general for any displacement δs, whose components are δx, δy, δz, we have $\underset{\cdot}{X}\,\delta x + \underset{\cdot}{Y}\,\delta y + \underset{\cdot}{Z}\,\delta z = \underset{\cdot}{R}\,\delta r$, where $\underset{\cdot}{R}$ is the impulse whose components are $\underset{\cdot}{X}$, $\underset{\cdot}{Y}$, $\underset{\cdot}{Z}$, and δr is the projection of δs on the direction of $\underset{\cdot}{R}$. Hence, if the directions of $\underset{\cdot}{R}$ and δs are at right angles to each other,

$$\underset{\cdot}{X}\,\delta x + \underset{\cdot}{Y}\,\delta y + \underset{\cdot}{Z}\,\delta z = 0.$$

Again, if two equal and opposite impulses occur at points whose coordinates are x_1, y_1, z_1; x_2, y_2, z_2, the corresponding terms in

$$\Sigma(\underset{\cdot}{X}\delta x + \underset{\cdot}{Y}\delta y + \underset{\cdot}{Z}\delta z)$$

are $$\underset{\cdot}{X}(\delta x_1 - \delta x_2) + \underset{\cdot}{Y}(\delta y_1 - \delta y_2) + \underset{\cdot}{Z}(\delta z_1 - \delta z_2),$$

or $\underset{\cdot}{R}(\delta r_1 - \delta r_2)$, from which we conclude, that if the relative displacement of two points be perpendicular to the direction of the mutual impulsive reaction at those points, the corresponding terms in $\Sigma(\underset{\cdot}{X}\delta x + \underset{\cdot}{Y}\delta y + \underset{\cdot}{Z}\delta z)$ vanish.

We can now prove the following theorems:—

If a system be acted on by external impulses, the *vis viva* is diminished by the *vis viva* of the additional motion when the impulse at each point is perpendicular to the *subsequent* velocity of that point, but increased by the same amount when the impulse is perpendicular to the *antecedent* velocity.

Similar results hold good for internal impulsive reactions when each mutual impulse is perpendicular to the relative velocity of the points between which it acts.

For the same notation being adopted as in Art. 197— 1° when each impulse is perpendicular to the *subsequent* velocity of the point at which it acts, we have

$$\Sigma(\underset{\cdot}{X}u + \underset{\cdot}{Y}v + \underset{\cdot}{Z}w) = 0;$$

and 2° when it is perpendicular to the *antecedent* velocity,

$$\Sigma(\underset{\cdot}{X}u' + \underset{\cdot}{Y}v' + \underset{\cdot}{Z}w' = 0.$$

In the first case, substituting u, v, w for $\delta x, \delta y, \delta z$ in equation (4), we get

$$\Sigma m\{(u - u')u + (v - v')v + (w - w')w\} = 0;$$

from which we obtain

$$\Sigma m\{(u - u')^2 + (v - v')^2 + (w - w')^2\}$$
$$= \Sigma m(u'^2 + v'^2 + w'^2) - \Sigma m(u^2 + v^2 + w^2). \qquad (13)$$

In the second case, substituting u', v', w' for $\delta x, \delta y, \delta z$ in (4), we obtain in like manner

$$\Sigma m\{(u - u')^2 + (v - v')^2 + (w - w')^2\}$$
$$= \Sigma m(u^2 + v^2 + w^2) - \Sigma m(u'^2 + v'^2 + w'^2). \qquad (14)$$

Bertrand's Theorem (Art. 199) is obviously included in the first case of the above theorems.

The impulses resulting from the impact of inelastic bodies against fixed obstacles, or against one another, as well as those produced by sudden pulls on inextensible strings, come under the first case considered above. To the second case, on the other hand, belong impulses due to explosions, or to the process of restitution which takes place in the second period of the impact of elastic bodies.

As has been already stated (Art. 78), in the impact of such bodies there are two periods. In the first, the *mutual action reduces the relative normal velocity of the colliding points to zero.* In the second, a force of restitution is developed, which acts at each point in the same direction as the original force, and produces an impulse which bears a constant ratio to that belonging to the first period. This constant ratio is called the coefficient of restitution.

As the special equations which determine the changes of velocity in terms of the corresponding impulses are obtained by equating to zero the coefficients of the independent variations in equation (4), Art. 197, we see that these equations are always linear. Moreover, in the impact of elastic bodies the geometrical conditions are the same in the periods of compression and of restitution; but each impulse in the latter period is equal to the corresponding impulse in the former multiplied by the coefficient of restitution. Hence we conclude that this holds good likewise for the corresponding changes of velocity in the two periods.

Examples.

1. If a system be acted on by any set of impulses, prove that the increase of its *vis viva* may be expressed in the form $\Sigma\{\underset{\cdot}{X}(u+u') + \underset{\cdot}{Y}(v+v') + \underset{\cdot}{Z}(w+w')\}$, where $\underset{\cdot}{X}$, $\underset{\cdot}{Y}$, $\underset{\cdot}{Z}$ are the components of the impulse acting at any point of the system; and u, v, w, u', v', w' the components of the velocity of this point after and before the impulsion, respectively.

2. If a system be acted on by a set of impulses which reduce to zero the velocities, in the direction of the impulses, of the points at which they act, find the change of *vis viva* in terms of the impulses and the antecedent velocities of the points at which they act. *Ans.* $\Sigma(\underset{\cdot}{X}u' + \underset{\cdot}{Y}v' + \underset{\cdot}{Z}w')$.

3. The ends of a string passing over a smooth pulley are attached to two masses, of which one rests on a horizontal plane, and the other is dropped through a height h, the masses of the string and pulley being neglected, determine the loss of kinetic energy caused by the impulsive tension of the string.

If m and m' be the masses, v_1 and v_2 the velocities of the dropped mass m before and after the chuck, and $\mathcal{J}$ the loss of kinetic energy,

$$2\mathcal{J} = m(v_1 - v_2)^2 + m' v_2^2. \quad \text{Hence } \mathcal{J} = \frac{mm'}{m+m'} gh.$$

4. If any system of smooth imperfectly elastic bodies having a common coefficient of restitution collide, show that the loss of *vis viva* is

$$\frac{1-e}{1+e} \Sigma m \{(u-u')^2 + (v-v')^2 + (w-w')^2\},$$

where e is the coefficient of restitution, m the mass of any particle, and u', v', w', u, v, w the components of its velocity before and after the shock.

Let U, V, W be the components of the velocity of m at the end of the first period of impact; then by equations (13) and (14), Art. 202, if $\mathcal{J}$ be the total loss of kinetic energy,

$$2\mathcal{J} = \Sigma m\{(U-u')^2 + (V-v')^2 + (W-w')^2\} - \Sigma m\{(u-U)^2 + (v-V)^2 + (w-W)^2\};$$

but $\quad u - U = e(U-u')$, &c., $\quad$ and therefore, $\quad u - u' = (1+e)(U-u')$, &c.

Hence, $\quad 2\mathcal{J} = (1-e^2)\Sigma m\{(U-u')^2 + (V-v')^2 + (W-w')^2\}$

$$= \frac{1-e^2}{(1+e)^2} \Sigma m\{(u-u')^2 + (v-v')^2 + (w-w')^2\}.$$

The theorem contained in this Example is due to Carnot.

5. Two weights are connected by a fine inextensible string passing over a smooth pulley. The lesser hangs vertically, and the other descends a smooth inclined plane, starting without initial velocity from a point vertically under the pulley; determine how far it will descend; and state the limit of the ratio of the weights within which finite descent is possible.

If z be the height which the lesser weight W' ascends, and s the distance along the inclined plane traversed by the greater weight W, we have, by the equation of *vis viva*, when the system comes to rest, $Ws \sin i - W'z = 0$, and therefore $z = \lambda s \sin i$, if $\frac{W}{W'} = \lambda$. Also, if h be the vertical distance of the pulley from the inclined plane, we have geometrically, since the string is inextensible,

$$(h+z)^2 = h^2 + s^2 + 2hs \sin i. \quad \text{Hence } s = \frac{2(\lambda - 1)\sin i}{1 - \lambda^2 \sin^2 i} h.$$

In order that finite descent should be possible, $W' > W \sin i$.

6. Two equal spheres, A and B, starting simultaneously from rest, descend down two equally inclined planes; the one plane quite smooth, the other perfectly rough; find the ratio of the velocities of the centres of the spheres at the end of any time.

Let v_1 be the velocity of the centre of A, v_2 that of the centre of B, and ω the angular velocity of B at the end of any time t, then $v_2 = a\omega$ (see Ex. 1,

Art. 134), also, $mv_1^2 = 2gmz_1$, and $\frac{7}{5}\ ma^2\omega^2 = 2gmz_2$, where z_1 and z_2 are the distances through which the centres of A and B have descended. Now, if s_1 and s_2 be the distances which the centres have moved parallel to the inclined plane at any time,

$$v_1 = \frac{ds_1}{dt}, \quad v_2 = \frac{ds_2}{dt}, \quad \text{and } z_1 = s_1 \sin i, \quad z_2 = s_2 \sin i.$$

Hence, substituting and differentiating, we have

$$\frac{d^2 s_1}{dt^2} = g \sin i, \quad \frac{7}{5}\frac{d^2 s_2}{dt^2} = g \sin i. \quad \text{Hence } v_2 = \frac{5}{7} v_1.$$

7. A thin uniform rod, AB, slides down between a vertical and a horizontal rod, to which it is attached by small smooth rings; find the angular velocity of AB in any position.

Take the horizontal and vertical rods as axes of x and y, their intersection being O, and let θ be the angle which AB makes with the vertical rod at any time; then M, the middle point of AB, describes a circle round O with an angular velocity $\dot\theta$, which is likewise the angular velocity of the rod round M. Hence (Art. 134), $ma^2\dot\theta^2 + m\frac{a^2}{3}\dot\theta^2 = 2gma(\cos\alpha - \cos\theta)$, where a is half the length of the rod, m its mass, and α the initial value of θ; then

$$\dot\theta^2 = \frac{3g}{2a}(\cos\alpha - \cos\theta).$$

8. A narrow smooth semicircular tube, whose radius is a, is fixed in a vertical plane, the vertex of the semicircle being its highest point; a heavy flexible string passing through the tube hangs at rest; if the string be cut at one end of the tube, find the velocity which the longer portion will have attained when leaving the tube.

Let l be the length of each of the portions of string which hang below the ends of the tube in the position of equilibrium; then, since the distance of the centre of inertia of a semicircular arc from the centre is $\frac{2a}{\pi}$, a being the radius, we have, if v be the velocity with which the string leaves the tube,

$$(l + \pi a)\, v^2 = 2g\left\{ l\pi a + \pi a\left(\frac{2a}{\pi} + \frac{\pi a}{2}\right)\right\},$$

whence

$$v^2 = \frac{2\pi l + (4 + \pi^2)\, a}{l + \pi a}\, ag.$$

9. A uniform bar, of length $2a$, is suspended from a fixed, parallel, and equal horizontal bar, by strings of equal length joining the adjacent extremities of the bars. An angular velocity ω is imparted to the snspended bar round a vertical axis through its centre of inertia. Determine the vertical height through which its centre of inertia will rise.

As each extremity of the bar moves on the surface of a sphere to which the attached string is radius, the tensions of the strings do not appear in the equation of *vis viva*; hence $h = \frac{a^2\omega^2}{6g}$.

203. **The General Equations of Motion of a Rigid Body apply to every System.**—If the forces acting on any system be in equilibrium, the equilibrium is not disturbed by rendering the mutual distances of the points of the system invariable—in other words, by making it *rigid*. Hence the equations of motion of a rigid body are, in their most general form, applicable to any system whatever. The special reductions which may be applied to the forces of inertia in the case of a rigid body cannot, however, be employed in other cases.

204. **Equations of Motion of a Rigid Body.**—By means of D'Alembert's Principle we can at once write down the equations of motion of a rigid body. We have, in fact, merely to write down the six equations of equilibrium, taking into acconnt, not only the applied forces, but also the forces of inertia as defined in Art. 196.

Hence the six equations of motion are—

$$\Sigma m \frac{d^2x}{dt^2} = \Sigma X, \quad \Sigma m \frac{d^2y}{dt^2} = \Sigma Y, \quad \Sigma m \frac{d^2z}{dt^2} = \Sigma Z. \tag{15}$$

$$\left.\begin{aligned} \Sigma m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) &= \Sigma(yZ - zY) = L \\ \Sigma m\left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right) &= \Sigma(zX - xZ) = M \\ \Sigma m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) &= \Sigma(xY - yX) = N \end{aligned}\right\}, \tag{16}$$

where L, M, N are the moments of the applied forces round the axes.

For impulses, the corresponding equations are—

$$\Sigma m(u - u') = \Sigma \underset{\cdot}{X}, \quad \Sigma m(v - v') = \Sigma \underset{\cdot}{Y}, \quad \Sigma m(w - w') = \Sigma \underset{\cdot}{Z}. \tag{17}$$

$$\left.\begin{aligned} \Sigma m\{y(w - w') - z(v - v')\} &= \Sigma(y\underset{\cdot}{Z} - z\underset{\cdot}{Y}) = \underset{\cdot}{L} \\ \Sigma m\{z(u - u') - x(w - w')\} &= \Sigma(z\underset{\cdot}{X} - x\underset{\cdot}{Z}) = \underset{\cdot}{M} \\ \Sigma m\{x(v - v') - y(u - u')\} &= \Sigma(x\underset{\cdot}{Y} - y\underset{\cdot}{X}) = \underset{\cdot}{N} \end{aligned}\right\}. \tag{18}$$

These equations hold good for any system which is

altogether free, *i. e.* unacted on by any constraints, and not subject to geometrical conditions external to itself.

In the case of a system subject to external constraints, the constraints must, in general, be replaced by the stresses to which they give rise.

Equations (15) and (17) may be put into a simpler shape as follows:—

205. **Motion of the Centre of Inertia of a Free System.**—Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the coordinates of the centre of inertia, and $\mathfrak{M}$ the mass of the entire system; then, for continuous forces,

$$\left.\begin{aligned} \mathfrak{M}\ddot{\bar{x}} &= \Sigma m \frac{d^2x}{dt^2} = \Sigma X \\ \mathfrak{M}\ddot{\bar{y}} &= \Sigma m \frac{d^2y}{dt^2} = \Sigma Y \\ \mathfrak{M}\ddot{\bar{z}} &= \Sigma m \frac{d^2z}{dt^2} = \Sigma Z \end{aligned}\right\}. \qquad (19)$$

Also, for impulses,

$$\left.\begin{aligned} \mathfrak{M}(\bar{u} - \bar{u}') &= \Sigma m(u - u') = \Sigma \underset{\cdot}{X} \\ \mathfrak{M}(\bar{v} - \bar{v}') &= \Sigma m(v - v') = \Sigma \underset{\cdot}{Y} \\ \mathfrak{M}(\bar{w} - \bar{w}') &= \Sigma m(w - w') = \Sigma \underset{\cdot}{Z} \end{aligned}\right\}. \qquad (20)$$

These equations give the motion of the centre of inertia of a free system acted on by any forces. From them it appears that—

The centre of inertia of a free system moves as if all the forces were applied to the entire mass concentrated there.

206. **Constraints and Partial Freedom.**—If a system be subject to external constraints, we may apply equations (19) and (20), provided we suppose the constraints replaced by the forces to which they give rise.

If a system, though not entirely free, be such that equal and parallel displacements of arbitrary magnitude can be given to each of its points in a definite direction, and if the axis of x be taken in that direction, we have still the equation

$$\mathfrak{M}\ddot{\bar{x}} = \Sigma X.$$

207. **Internal Forces.**—Any force by which two parts of a system act on each other is said to be internal.

Since action and reaction are equal and opposite, the components of internal forces destroy one another in the sums ΣX, ΣY, and ΣZ. Hence in any system

Internal forces, whether continuous or impulsive, have no effect on the motion of the centre of inertia.

208. **Case of no External Forces.**—It follows from Art. 207, that if a system be acted on by no external forces, its centre of inertia is either at rest or moves in a straight line with a constant velocity. This theorem is sometimes termed *The Principle of the Conservation of the Motion of the Centre of Inertia.*

Results similar to those of the preceding Articles hold good for impulses.

209. **Motion of a Free System relative to its Centre of Inertia.**—If ξ, η, ζ be the coordinates of any point of a system referred to axes through its centre of inertia parallel to fixed directions, $x = \bar{x} + \xi$, $y = \bar{y} + \eta$, $z = \bar{z} + \zeta$. Substituting in D'Alembert's equation, we have

$$\delta\bar{x}\left(\Sigma X - (\Sigma m)\frac{d^2\bar{x}}{dt^2}\right) + \delta\bar{y}\left(\Sigma Y - (\Sigma m)\frac{d^2\bar{y}}{dt^2}\right) + \delta\bar{z}\left(\Sigma Z - (\Sigma m)\frac{d^2\bar{z}}{dt^2}\right)$$

$$- \delta\bar{x}\,\Sigma m\frac{d^2\xi}{dt^2} - \delta\bar{y}\,\Sigma m\frac{d^2\eta}{dt^2} - \delta\bar{z}\,\Sigma m\frac{d^2\zeta}{dt^2}$$

$$- \frac{d^2\bar{x}}{dt^2}\Sigma m\,\delta\xi - \frac{d^2\bar{y}}{dt^2}\Sigma m\,\delta\eta - \frac{d^2\bar{z}}{dt^2}\Sigma m\,\delta\zeta$$

$$+ \Sigma\left\{\left(X - m\frac{d^2\xi}{dt^2}\right)\delta\xi + \left(Y - m\frac{d^2\eta}{dt^2}\right)\delta\eta + \left(Z - m\frac{d^2\zeta}{dt^2}\right)\delta\zeta\right\} = 0.$$

Now $\Sigma m\xi = \Sigma m\eta = \Sigma m\zeta = 0$; hence, by equations (19), Art. 205, we obtain

$$\Sigma\left\{\left(X - m\frac{d^2\xi}{dt^2}\right)\delta\xi + \left(Y - m\frac{d^2\eta}{dt^2}\right)\delta\eta + \left(Z - m\frac{d^2\zeta}{dt^2}\right)\delta\zeta\right\} = 0. \quad (21)$$

It follows from this equation that—

The motion of a free system relative to its centre of inertia is the same as if this point were fixed in space, the applied forces

being unaltered as regards magnitude, direction, and point of application.

The theorem just stated holds good as well for impulsive as for continuous forces. This readily appears by applying the transformation employed above to equation (4), Art. 197, and making use of equations (20), Art. 205.

210. **Moments of Momentum.**—It is readily seen that at any instant the expression

$$\Sigma m(xv - yu) \text{ or } \Sigma m(x\dot{y} - y\dot{x})$$

represents the entire moment of the momenta round the axis of z of all the elements of the system at the instant: and similarly $\Sigma m(y\dot{z} - z\dot{y})$ and $\Sigma m(z\dot{x} - x\dot{z})$ represent the corresponding quantities relative to the axes of x and y, respectively.

These moments of momenta are of fundamental importance in the discussion of the motion of any system, and we shall accordingly represent them by distinct symbols.

Thus let

$$H_1 = \Sigma m(y\dot{z} - z\dot{y}),\ H_2 = \Sigma m(z\dot{x} - x\dot{z}),\ H_3 = \Sigma m(x\dot{y} - y\dot{x}), \quad (22)$$

then equations (18) may be written in the form;

$$H_1 - H_1' = \underset{\cdot}{L}, \quad H_2 - H_2' = \underset{\cdot}{M}, \quad H_3 - H_3' = \underset{\cdot}{N}, \quad (23)$$

in which H_1', H_2', H_3' represent the moments of momenta of the system before, and H_1, H_2, H_3 those after, the impact.

If the body be at rest when acted on by the impulses, these equations become

$$H_1 = \underset{\cdot}{L}, \quad H_2 = \underset{\cdot}{M}, \quad H_3 = \underset{\cdot}{N}. \quad (24)$$

Hence, in this case, the moments of momenta generated by the impulses are respectively equal to the impulsive moments applied.

Next, since $$x\ddot{y} - y\ddot{x} = \frac{d}{dt}(x\dot{y} - y\dot{x}),$$

we have $$\frac{dH_1}{dt} = \Sigma m(x\ddot{y} - y\ddot{x}),$$

and it follows that equations (16) may be expressed in the following form :—

$$\frac{dH_1}{dt} = L, \quad \frac{dH_2}{dt} = M, \quad \frac{dH_3}{dt} = N. \qquad (25)$$

The quantities H_1, H_2, H_3 admit of an important transformation, as follows :—

If $\frac{1}{2} h_3\, dt$ represent the elementary area described round the origin by the projection of the point xyz on the plane of xy, then

$$x\dot{y} - y\dot{x} = h_3.$$

Hence $H_3 = \Sigma m h_3$, and, likewise, representing the projections on the planes of yz and xz by a similar notation,

$$H_1 = \Sigma m h_1, \quad H_2 = \Sigma m h_2.$$

Accordingly equations (25) may be written

$$\Sigma m \frac{dh_1}{dt} = L, \quad \Sigma m \frac{dh_2}{dt} = M, \quad \Sigma m \frac{dh_3}{dt} = N. \qquad (26)$$

The corresponding equations for impulses are

$$\Sigma m h_1 = \underset{.}{L}, \quad \Sigma m h_2 = \underset{.}{M}, \quad \Sigma m h_3 = \underset{.}{N}. \qquad (27)$$

If the system is in motion when the impulses act, the three latter equations should be written

$$\Sigma m h_1 = \underset{.}{L} + \Sigma m h_1', \quad \Sigma m h_2 = \underset{.}{M} + \Sigma m h_2', \quad \Sigma m h_3 = \underset{.}{N} + \Sigma m h_3', \qquad (28)$$

where h_1', h_2', h_3' are the values of h_1, h_2, h_3 the instant before the impulses act.

The quantities h_1, h_2, h_3, &c., are the areal velocities, relative to the origin, of the different points of the system; and $\frac{dh_1}{dt}$, &c., are the areal accelerations (*see* Art. 29).

In any system in motion the three moments H_1, H_2, H_3, if they were regarded as moments of forces or couples acting on the same rigid body, would be equivalent to a

single moment H round a line whose direction cosines are $\frac{H_1}{H}$, $\frac{H_2}{H}$, $\frac{H_3}{H}$; H being given by the equation

$$H^2 = H_1^2 + H_2^2 + H_3^2.$$

This line is called the *momentum axis* of the system relative to the origin. As it is the axis of the couple which is the resultant of the couples corresponding to the moments of the momenta of the different elements of the system, it is plain that its direction is independent of the directions of the coordinate axes.

If Sdt be twice the sum of the projections of the elementary areas described by all the points of the system round the origin, each multiplied by the corresponding element of mass, on a plane whose normal makes an angle θ with the momentum axis, then

$$S = H \cos \theta. \tag{29}$$

This may be proved in the following manner: Let hdt be double the elementary area described by the element whose mass is m round the origin; and let a, β, γ be the cosines of the angles its plane makes with the coordinate planes; then, λ, μ, ν being the direction cosines of the normal to the plane of S,

$$S = \Sigma mh\,(a\lambda + \beta\mu + \gamma\nu) = \lambda\Sigma mha + \mu\Sigma mh\beta + \nu\Sigma mh\gamma$$

$$= \lambda H_1 + \mu H_2 + \nu H_3 = H\left\{\lambda\frac{H_1}{H} + \mu\frac{H_2}{H} + \nu\frac{H_3}{H}\right\} = H\cos\theta.$$

Hence, *the multiple sum of the projections of the elementary areas on the plane at right angles to the momentum axis is a maximum.*

This plane is called the *Principal Plane* relative to the origin. From what has been just proved, we see again that its position is independent of the directions of the axes.

If ξ, η, ζ be a second set of rectangular axes through the origin parallel to directions fixed in space, and if the direc-

tion cosines of ξ, referred to x, y, z, be a_1, a_2, a_3; of η, b_1, b_2, b_3; of ζ, c_1, c_2, c_3; we have, as particular cases of what has been proved above,

$$\left.\begin{aligned}\Sigma m(\eta\dot{\zeta} - \zeta\dot{\eta}) &= H_1 a_1 + H_2 a_2 + H_3 a_3 \\ \Sigma m(\zeta\dot{\xi} - \xi\dot{\zeta}) &= H_1 b_1 + H_2 b_2 + H_3 b_3 \\ \Sigma m(\xi\dot{\eta} - \eta\dot{\xi}) &= H_1 c_1 + H_2 c_2 + H_3 c_3\end{aligned}\right\}. \qquad (30)$$

The preceding theorems of this Article are true for any system of moving points, and whether the origin be fixed or movable.

Again, *to find the moments of momentum of a system round axes intersecting at a point whose coordinates are a, b, c.*

Let H_1', H_2', H_3' be the moments of momentum of the system round axes parallel to the coordinate axes, and intersecting at the point abc; then, we have

$$\begin{aligned}H_1' &= \Sigma m\{(y - b)\dot{z} - (z - c)\dot{y}\} \\ &= \Sigma m(y\dot{z} - z\dot{y}) - (b\Sigma m\dot{z} - c\Sigma m\dot{y});\end{aligned}$$

but $\bar{x}$, $\bar{y}$, $\bar{z}$ being the coordinates of the centre of inertia of the system,

$$\Sigma m\dot{y} = \mathfrak{M}\dot{\bar{y}}, \quad \Sigma m\dot{z} = \mathfrak{M}\dot{\bar{z}};$$

hence we obtain

$$\left.\begin{aligned}H_1' &= H_1 - \mathfrak{M}(b\dot{\bar{z}} - c\dot{\bar{y}}) \\ H_2' &= H_2 - \mathfrak{M}(c\dot{\bar{x}} - a\dot{\bar{z}}) \\ H_3' &= H_3 - \mathfrak{M}(a\dot{\bar{y}} - b\dot{\bar{x}})\end{aligned}\right\}. \qquad (31)$$

Again, *the moment of momentum of a system round an axis, through any point O, is equal to the moment of the momentum relative to the centre of inertia round a parallel axis through that point, together with the moment of momentum round the axis through O of the entire mass of the system supposed to be concentrated at the centre of inertia, and moving with it.*

Take the axis through O as the axis of x, and make use of the transformation employed in Art. 209, then

$$\Sigma m\,(y\dot{z} - z\dot{y}) = \Sigma m\,\{(\bar{y} + \eta)(\dot{\bar{z}} + \dot{\zeta}) - (\bar{z} + \zeta)(\dot{\bar{y}} + \dot{\eta})\}$$

$$= \mathfrak{M}\,(\bar{y}\dot{\bar{z}} - \bar{z}\dot{\bar{y}}) + \Sigma m\,(\eta\dot{\zeta} - \zeta\dot{\eta})\,; \qquad (32)$$

since $\qquad \Sigma m\eta = \Sigma m\zeta = 0, \quad \Sigma m\dot{\eta} = \Sigma m\dot{\zeta} = 0.$

The student will observe that $\dot{\xi}$, $\dot{\eta}$, &c., denote *relative*, not absolute, velocities. If the origin O be the centre of inertia of the system, equations (23), (24), and (25) hold good whether O be fixed or moving (Art. 209), the axes being parallel to lines fixed in space.

In the deduction of equations (23), (24), and (25), we have supposed that the system is free, that is, unacted on by constraints external to the system itself.

211. **Constraints and Partial Freedom.**—A system which is not free may be regarded as free, if the external constraints be replaced by the forces to which they give rise.

In general, we can ascertain whether a given constraint affects the validity of equations (23), (24), and (25), by considering its influence on the conditions of equilibrium of a rigid body.

If one point of a rigid body be fixed, we know that for its equilibrium the moments of the applied forces round three rectangular axes meeting at the point must each be equal to zero. Hence we conclude—

If there be one point of a system fixed, equations (23), (24), *and* (25) *hold good for this point as origin.*

Again, if there be a fixed line in a rigid body, the condition of equilibrium is that the moment of the applied forces round this line should be zero. From this we infer—

If there be a fixed line in a system, the rate of change relative to the time in the moment of momentum of the system round this line is equal to the moment of the applied forces.

212. **Internal Forces.**—Since internal forces occur in pairs, each pair consisting of two equal and opposite forces

having a common line of direction, the moment round any line of the whole set of internal forces must be zero. Hence *the moments of momentum of any system are unaffected by forces internal to the system.*

213. **Conservation of Moment of Momentum.**—If a free system be unacted on by any forces external to itself, its resultant moment of momentum, relative to any point fixed in space, is constant, and has for its axis a line whose direction is invariable.

A similar result holds good for the centre of inertia even though this point be not fixed in space.

If a system, otherwise free, contain a point or a line fixed in space, and be unacted on by external forces, the resultant moment of momentum of the system relative to the fixed point, or the moment of momentum round the fixed line, is constant.

The theorems enunciated in this Article together constitute what has been often termed *The Principle of the Conservation of the Moment of Momentum*, or *The Principle of the Conservation of Areas.*

As the moment of a force round an axis intersecting the line of direction of the force is zero, we see that—

If the lines of direction of all the external forces which act on a free system be met by the same space axis, the moment of momentum of the system round this axis is constant.

If the space-axis be fixed in the system, which is otherwise free, the theorem above still holds good.

In a similar manner we may conclude that—

If a system receive an impulse, the moment of momentum of the system round an axis fixed in space, and passing through any point on the line of direction of the impulse, remains the same as before.

Examples.

1. In any system in motion, show that the moments of momenta round three rectangular axes are equal to the moments of the impulses which would impart to the system if at rest its actual motion.

2. If ξ, η, ζ be the coordinates, relative to the centre of inertia, of any point of a free system, show directly, that if the system start from rest,

$$\Sigma m (\eta\dot{\zeta} - \zeta\dot{\eta}) = \Sigma (\eta \underset{\cdot}{Z} - \zeta \underset{\cdot}{Y}), \text{ \&c.},$$

and that during the motion,

$$\Sigma m(\eta\ddot{\zeta} - \zeta\ddot{\eta}) = \Sigma(\eta Z - \zeta Y), \text{ \&c.}$$

By equation (32), Art. 210, we have

$$\mathfrak{M}(\bar{y}\dot{\bar{z}} - \bar{z}\dot{\bar{y}}) + \Sigma m(\eta\dot{\zeta} - \zeta\dot{\eta}) = \Sigma m(y\dot{z} - z\dot{y}) = \underset{\cdot}{L} = \Sigma\{(\bar{y} + \eta)\underset{\cdot}{Z} - (\bar{z} + \zeta)\underset{\cdot}{Y}\};$$

but $\mathfrak{M}\dot{\bar{y}} = \Sigma\underset{\cdot}{Y}$, and $\mathfrak{M}\dot{\bar{z}} = \Sigma\underset{\cdot}{Z}$; therefore, &c.

Again, differentiating each side of the equation (32) of Art. 210, we have

$$\mathfrak{M}(\bar{y}\ddot{\bar{z}} - \bar{z}\ddot{\bar{y}}) + \Sigma m(\eta\ddot{\zeta} - \zeta\ddot{\eta}) = \Sigma m(y\ddot{z} - z\ddot{y}) = L = \Sigma\{(y + \eta)Z - (z + \zeta)Y\};$$

and as $\mathfrak{M}\ddot{\bar{y}} = \Sigma Y$, $\mathfrak{M}\ddot{\bar{z}} = \Sigma Z$,

we obtain the required result.

3. A satellite of mass m is moving in a circle whose radius is r, round a planet whose mass is M, and which rotates round an axis perpendicular to the plane of the orbit with an angular velocity n. If C be the moment of inertia of the planet, and μ the attraction between unit masses at the unit of distance, show that the moment of momentum of the system round its centre of inertia is

$$C\left\{n + \mu^{\frac{1}{2}}\frac{Mm}{C}(M + m)^{-\frac{1}{2}}r^{\frac{1}{2}}\right\}.$$

4. A heavy particle moves on a smooth surface of revolution whose axis is vertical; prove that the moment of momentum of the particle round the axis is constant.

5. A number of mutually attracting particles are acted on by forces passing through the same fixed point; prove that their resultant moment of momentum relative to this point is constant, and that the direction of its axis is invariable.

6. A system is acted on by no external force except gravity; prove that its moments of momenta round axes parallel to fixed directions in space, and intersecting at its centre of inertia, are constant.

7. Show that the centre of inertia of the universe is either fixed in space or else moves in a straight line with a constant velocity.

8. A man walks from one end to the other of a uniform plank which is placed on a smooth horizontal table; determine the displacement of the plank.

Let a be the length of the plank, P its mass, M that of the man; the displacement is $\dfrac{M}{M + P}a$.

9. A uniform plank is placed on a smooth inclined plane, so as to be perpendicular to the intersection of the inclined plane with the horizon; determine the

time in which a man should go from the upper to the lower end of the plank in order that it should remain unmoved.

Let t be the time required. The displacement of the centre of inertia of the system in the time t in space is $\frac{1}{2}gt^2 \sin i$, and relative to the plank is $\frac{M}{M+P}a$. If the plank remain unmoved these must be equal. Hence

$$t^2 = \frac{2M}{M+P} \cdot \frac{a}{g \sin i}.$$

10. The base of a smooth homogeneous circular semi-cylinder rests on a horizontal plane. A particle m is placed at a point on the surface of the semi-cylinder, situated in a vertical plane containing its centre of inertia and perpendicular to its axis. Show that the particle will describe an ellipse.

Let the axis of x be the intersection of the vertical plane, in which the particle moves, with the horizontal plane on which the semi-cylinder rests; the axis of y being vertical. Let x, y be the coordinates of the particle, x' the coordinate of the centre of inertia of the semi-cylinder, m' its mass, and a its radius.

Considering the whole system as one body, we have (Art. 206),

$$m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2} = 0.$$

Hence, since the system starts from rest, $mx + m'x'$ is constant, or the projection on the horizontal plane of the centre of inertia of the whole system remains fixed in space. Taking this point for origin, we have $mx + m'x' = 0$.

Again, since the semi-cylinder is homogeneous, we have, from the geometrical conditions,

$$(x - x')^2 + y^2 = a^2.$$

Substituting for x', we obtain

$$(m + m')^2 x^2 + m'^2 y^2 = m'^2 a^2.$$

11. Two particles, connected by a rigid rod whose weight is negligible, are projected along a smooth horizontal plane; determine their motion.

The position of the centre of inertia at any time is given by the equations

$$x = mt + a, \quad y = nt + b,$$

and the inclination of the rod to the axis of x by the equation $\theta = \omega t + \epsilon$, where m, n, a, b, ω, and ϵ are constants.

12. Two equal particles are connected together by a fine inextensible string; one of them is placed on a smooth table, the other just over the edge, the string being at full stretch at right angles to the edge; find the interval of time from the instant at which the particle originally on the table leaves it to the instant at which the string first becomes horizontal.

The acceleration of the particle moving on the table is $\frac{1}{2}g$. Hence, if c be the length of the string, the particle leaves the table with a horizontal velocity v, where $v^2 = gc$. At this instant the middle point of the string has a horizontal velocity $\frac{1}{2}v$, and the lower particle has no horizontal velocity. Hence

the moment of momentum of the system round a horizontal axis through the centre of inertia is $\frac{1}{2}mcv$. This remains constant (Ex. 6), and therefore twice the area described round the centre of inertia in any time t is $\frac{1}{2}mcvt$. If t be the interval of time during which the string passes from a vertical to a horizontal position, we have, therefore, $\frac{1}{4}\pi c^2 = \frac{1}{2}cvt$, and substituting for v its value, we obtain

$$t = \frac{1}{2}\pi\sqrt{\frac{c}{g}}.$$

13. A sphere is projected with a velocity v along a uniform smooth tube within which it fits exactly. The tube rests on a smooth horizontal plane, and its axis forms a circle; determine the motion.

Let m be the mass of the sphere, m' that of the tube, and a the radius of the circle formed by its axis. The common centre of inertia O of the tube and sphere moves parallel to the direction of projection of the sphere with a velocity $\frac{mv}{m+m'}$, and the centres of the tube and sphere describe circles round O with an angular velocity $\frac{v}{a}$.

14. A spherical shell rests upon a smooth horizontal plane; a particle is placed at the lowest point of the internal surface of the shell, which is then projected with a horizontal velocity V. The internal surface of the shell being smooth, determine to what height the particle will ascend.

Let x and y be the coordinates of the particle, m its mass, and v its velocity; x' and y' the coordinates, and v' the velocity of the centre of the shell, m' being its mass. Take as axis of x the intersection of the smooth horizontal plane with the vertical plane of motion; then, Art. 200,

$$mv^2 + m'v'^2 = m'V^2 - 2mgy,$$

and, by Art. 206,

$$m\dot{x} + m'\dot{x}' = m'V.$$

Also, as the particle remains on the sphere whose radius is a, we have

$$(x - x')^2 + (y - y')^2 = a^2;$$

whence, differentiating, and remembering that $\dot{y}' = 0$, we have $\dot{x} - \dot{x}' = 0$ when $\dot{y} = 0$. Hence, substituting, we obtain

$$y = \frac{m'}{2(m+m')}\frac{V^2}{g}.$$

This result may not hold good if the value of y given above exceed a.

15. A smooth tube, movable in a horizontal plane about a vertical axis, is charged with a number of balls at given intervals; an angular velocity Ω is communicated to the tube; determine the velocities of the tube and of the balls at any assigned distances of the latter from the axis.

Let m_1, m_2, &c. be the masses of the balls, a_1, a_2 &c. their initial distances

from the axis, r_1, r_2, &c. their distances at any instant, ω the angular velocity, and Mk^2 the moment of inertia of the tube about the axis; then (Arts. 213, 200),

$$(m_1 r_1^2 + m_2 r_2^2 + \&c. + Mk^2)\,\omega = (m_1 a_1^2 + m_2 a_2^2 + \&c. + Mk^2)\,\Omega,$$

$$m_1 \dot{r}_1^2 + m_2 \dot{r}_2^2 + \&c. + (m_1 r_1^2 + m_2 r_2^2 + \&c. + Mk^2)\,\omega^2 = (m_1 a_1^2 + m_2 a_2^2 + \&c. + Mk^2)\,\Omega^2.$$

Again (Art. 28), $$\frac{d^2 r_1}{dt^2} - r_1\omega^2 = 0 = \frac{d^2 r_2}{dt^2} - r_2\omega^2,$$

whence $$r_2 \frac{d^2 r_1}{dt^2} - r_1 \frac{d^2 r^2}{dt^2} = 0,$$

and integrating,

$$r_2 \frac{dr_1}{dt} - r_1 \frac{dr_2}{dt} = \text{constant} = 0.$$

Hence we have

$$\frac{r_1}{r_2} = \text{constant} = \frac{a_1}{a_2},$$

and therefore also $\frac{\dot{r}_1}{\dot{r}_2} = \frac{a_1}{a_2}$, with similar equations for the other distances and velocities. Substituting in the equations of momentum and vis viva, and putting

$$m_1 a_1^2 + m_2 a_2^2 + \&c. = I, \quad Mk^2 = I',$$

we obtain

$$(Ir_1^2 + I'a_1^2)\,\omega = (I + I')\,a_1^2\,\Omega, \quad (Ir_1^2 + I'a_1^2)\,\dot{r}_1^2 = (I + I')\,a_1^2(r_1^2 - a_1^2)\,\Omega^2, \ \&c.$$

16. An indefinitely great number of thin cylindrical shells are revolving in the same direction about their common axis, the angular velocity of each shell being proportional to a positive power of its radius. If the system of shells be suddenly united into a solid cylinder, find the angular velocity of the cylinder about its axis.

Let ω be the angular velocity of any shell, r its radius, Ω and R being those of the outermost shell, then $\omega = \lambda r^n$, and before the shells are united, the moment of momentum of the system is

$$\mu\, 2\pi\lambda \int_0^R r^{n+3}\,dr.$$

If ω' be the angular velocity of the united system, its moment of momentum is $\mu \frac{\pi R^4}{2}\omega'$. Equating these two, we obtain

$$\omega' = \frac{4\Omega}{n+4}.$$

17. A uniform horizontal stick falling to the ground strikes at one end against a stone; compare the blow it receives with what it would have received had both ends struck simultaneously against two stones, the blows being supposed to be perpendicular to the stick.

Let v' and v be the velocities of the middle point C of the stick, before and after it receives the single blow at the extremity A; let $2a$ be the length of the stick, m its mass, and P the impulse of the blow. The moment of momentum of the stick round a horizontal space axis through A remains unaltered by the blow. Before the blow the whole moment of momentum is due ((32), Art. 210) to the motion of the centre of inertia, the stick having no motion relative to it. After the blow the stick is rotating round A (since this point is reduced to rest) with an angular velocity ω. Hence $\frac{4}{3} ma^2 \omega = mav'$; but $v = a\omega$, and therefore, substituting, we have

$$v = \tfrac{3}{4} v', \quad \text{and} \quad v' - v = \tfrac{1}{4} v'.$$

Again, from the motion of the centre of inertia C, we obtain

$$P = m(v - v') = -\tfrac{1}{4} mv'.$$

In the second case, when the stick receives two blows each equal to Q, it is reduced to rest, and therefore

$$2Q = -mv', \quad \text{or} \quad Q = -\tfrac{1}{2} mv'.$$

Therefore finally $$P = \tfrac{1}{2} Q.$$

If the stick be elastic, the above investigation holds good for the impulses received during the first period of each impact; and as the total impulses are in a constant ratio to the former, the result is unaffected by the elasticity of the stick.

CHAPTER X.

MOTION OF A RIGID BODY PARALLEL TO A FIXED PLANE.

SECTION I.—*Kinematics.*

214. **Rigid Body, Determination of its Position.**—A body is said to be *rigid* when its constitution is such that the relative position of its points with respect to each other is unalterable.

The position of a point in space is usually determined by means of three rectangular coordinates, and depends therefore upon three independent quantities. It is easy to see that the position of a rigid body is determined by six independent variables. For, the position in space of a definite point A of the body is determined by three independent variables; two more are required to determine the plane in space in which a definite plane a of the body passing through A should lie; and finally, one more is necessary to fix in this plane a definite line of the body passing through A, and lying in the plane a. When the position in space of every point of the plane a is determined, it is obvious that the positions of all points of the rigid body are completely determined, since perpendiculars from them on the plane a are invariable in magnitude.

215. **Degrees of Freedom.**—As six independent quantities are required to determine the position of a rigid body, such a body, if subject to no restraint, is said to have six *degrees of freedom.*

It is plain, from what has been said, that if the positions of three points of a rigid body not lying on the same straight line are fixed, the position of every point of the body is determined.

216. **Motion of Translation.**—When a body moves so that the elements of the paths described by its different points are equal and parallel straight lines, the motion is said to be one of *translation.*

The path described by any one point of a body is, in general, a curve, and it appears from the above definition that the curves described by the different points during any motion of translation are equal and similar. Hence—

In a motion of translation, the line joining any two definite points of the body remains parallel to its initial position.

As the distances traversed by each point of the body are the same both in magnitude and direction, we may speak of *the motion of translation of the body*, and may compound any number of elementary motions in the same manner as for a point.

217. **Motion of Rotation.**—As already stated in Art. 95, when a body is moving in such a manner that each point is describing the arc of a circle having its centre on a fixed straight line, to which its plane is perpendicular, the motion is said to be a *rotation*, and the fixed straight line passing through the centres of all the circles is called the *axis of rotation.*

In a motion of this kind every point of the body lying on the axis of rotation remains fixed during the motion.

All lines in the body perpendicular to the axis of rotation turn through the same angle, which is called *the angular rotation*, or simply *the rotation* of the body.

Any line AB of the body which lies in a plane at right angles to the axis of rotation makes, at the end of the motion, an angle with its initial position, which is equal to the angular rotation of the body.

This readily appears as follows :—Join A to the point O in which the axis meets the plane in which AB lies; then, A' and B' being the new positions taken by A and B, since OA makes the same angle with AB which OA' makes with $A'B'$, the quadrilateral formed by OA, OA', AB and $A'B'$ can be inscribed in a circle, and therefore the angle between $A'B'$ and AB is equal to that between OA' and OA.

It is easy to see, that if two positions of a body have a straight line of particles in common, the body can be moved from one of these positions to the other by a rotation round this line.

218. **Motion Parallel to a Fixed Plane.**—When the paths described by the several points of a body during its motion are made up of elements, each of which is parallel to the same fixed plane, the motion of the body is said to be parallel to this plane.

If we consider any definite plane section of the body, which at the beginning of the motion is parallel to the fixed plane, this section must continue in the same plane throughout the motion, and its position at each instant determines the position of every point of the body. In order, therefore, to study the motion of a body moving parallel to a fixed plane, we have merely to investigate the motion of a plane figure in its own plane.

219. **Motion of a Plane Figure in its own Plane.**—A plane figure can be moved from any one position in its own plane to any other by giving it first a motion of translation, whereby any arbitrary point A of the figure is moved from its first position A_1 in space to its second position A_2, and then a motion of rotation round a perpendicular axis passing through A_2, whereby a definite line AB of the figure is moved into its final position in space A_2B_2. As the point A is perfectly arbitrary, the motion may be effected in an infinite variety of ways. The motion of translation to be given to the figure differs in general according as different points of the figure are chosen for A, but *the motion of rotation is in all cases the same.*

This readily appears from Art. 217, as the initial and final positions of any definite line of the figure are given, and the angle between them is in all cases the rotation of the body.

The results arrived at above depend upon the fact that the position of a plane figure in its own plane is completely determined by the position of one definite straight line of the figure. Hence also it appears that by properly selecting the point A, the motion of translation may in general be dispensed with altogether, or, in other words (*Differential Calculus*, Art. 293)—

A plane figure can be moved from any one position into any other in its own plane by a rotation round a point fixed in the plane.

In fact, BC being the original position of any definite line of the figure, and $B'C'$ its new position; if we join BB', bisect it, and erect a perpendicular, and do the same with CC', these two perpendiculars will, in general, determine by their intersection a point O, a rotation round which effects the given change of position.

If BB' be parallel to CC' this construction fails. Two cases then arise, according as BB' is equal to CC' or not. In the latter case, the intersection of BC and $B'C'$ is the centre of rotation. In the former the motion is one of pure translation, and the point O is at infinity.

As a particular case, it follows that—

Two rotations effected successively round two parallel axes bring a body into the same position as a single rotation round an axis parallel to the two former, the single rotation being equal in magnitude to the sum of the two to which it is equivalent.

We see also that—

A rotation round any given axis brings a body into the same position as an equal rotation round a parallel axis through any arbitrary point, together with a motion of translation.

Hence it appears that—

Equal and opposite rotations effected successively round two parallel axes A and B are equivalent to a single motion of translation.

For, a rotation round A is equivalent to an equal rotation round B, together with a translation; therefore equal and opposite rotations round A and B are equivalent to equal and opposite rotations round B, together with a translation; but equal and opposite rotations round B destroy each other; therefore, &c.

220. **Composition of Velocities.**—Hitherto we have been considering displacements having a finite magnitude. In regard to such displacements the order in which the several motions are effected is of importance, and in order to arrive at definite results it is necessary to specify whether the successive axes of rotation are fixed in space or in the body. In Kinetics, we are for the most part concerned not only with the initial and final positions of a body, but also with the

mode in which it passes from the one position to the other. It becomes then necessary to consider the infinitely small motions through which the body assumes its successive positions. Displacements effected in the same element of time divided by that element then become velocities, and the composition and equivalence of infinitely small displacements is tantamount to the composition and equivalence of velocities.

If the displacements received by a body be infinitely small, it is indifferent in what order rotations are effected round different axes fixed in space.

For, the changes produced in the coordinates of any point of the body by such a rotation are functions of its amplitude, and of the initial values of those coordinates. In the present case these changes are infinitely small, and therefore alterations in them, due to a previous displacement which is itself infinitely small, are infinitely small quantities of the second order.

Again, from similar considerations, it appears that *it is indifferent whether the axes be fixed in space or axes fixed in the body, whose positions at the commencement of the infinitely small motion coincide with those of the axes fixed in space.*

When the order of two displacements is indifferent they may be regarded as simultaneous, and if the resultant displacement be such as to move the body from one position into the next successive position, it is the actual motion of the body.

221. **Motion of a Rigid Body.**—The theorems of Article 219, when applied to infinitely small motions of a rigid body parallel to a fixed plane, give the following results:—

(1). The motion of a body parallel to a fixed plane consists at any instant of a velocity of rotation ω round an axis, passing through any arbitrary point A of the body, at right angles to the plane, and a velocity of translation v parallel to the plane.

(2). At each instant there is an axis, called the instantaneous axis of rotation, which is such that a velocity of rotation ω round it represents the whole motion of the body.

If r be the distance from A to this axis, and v the velocity of translation which the body must be considered to possess

when the axis of rotation is regarded as passing through A, then, in order that the axis should be at rest at the instant, the direction of v must be at right angles to r, and we must have $v = r\omega$. We thus see, that in a rigid body a motion of rotation together with a motion of translation, in a direction perpendicular to the axis of rotation, can be compounded into a motion which is one of rotation solely. Also, such motions cannot be compounded into a single rotation unless the direction of the translation is perpendicular to the axis of rotation.

(3). Two coexisting velocities of rotation round parallel axes are equivalent to a single velocity of rotation, equal to their sum, round an axis parallel to the two former, and dividing the distance between them inversely as the component velocities.

(4). Two equal and opposite velocities of rotation whose common magnitude is ω, round parallel axes, are equivalent to a velocity of translation, perpendicular to the plane of the axes, whose magnitude is $a\omega$, where a is the distance between the axes.

We see from what is stated above that velocities of rotation round parallel axes are compounded like parallel forces.

In considering rotations round parallel axes it is necessary to take into account not only the magnitudes of the rotations, but also their algebraical signs, or directions. The axis of rotation is supposed to be drawn from the feet of the spectator to his head, so that in estimating rotations the axis points towards the spectator. If two rotations round parallel axes, viewed from the same side of the plane perpendicular to the axes, are both in the same direction, they have like algebraical signs. The positive direction of rotation is of course arbitrary; but in the application of Analytic Geometry to rotational displacements it is usual to regard rotations as positive which bring a point—from the axis of X positive to the axis of Y positive, from Y positive to Z positive, and from Z positive to X positive. It follows from this, that if the axes of X and Y be drawn in the usual manner, a rotation opposite in direction to that of the hands of a watch is to be regarded as positive (Art. 87). Such a rotation is termed counter-clockwise.

222. **Analytical Treatment of the Motion of a Plane Figure in its own Plane.**—When a point moves in a circle whose centre is the origin, we may assume

$$x = r\cos\psi, \quad y = r\sin\psi,$$

whence $$\frac{dx}{dt} = -r\sin\psi\frac{d\psi}{dt}, \quad \frac{dy}{dt} = r\cos\psi\frac{d\psi}{dt},$$

and putting $\frac{d\psi}{dt} = \omega$, $\frac{dx}{dt} = \dot{x}$, $\frac{dy}{dt} = \dot{y}$, we have

$$\dot{x} = -y\omega, \quad \dot{y} = x\omega, \tag{1}$$

for the rotation of the point xy round the origin.

Now let x', y' be the coordinates of a definite point of a plane figure referred to axes fixed in space; x, y those of any point of the figure referred to the same axes; ξ, η those of the same point of the figure referred to axes *fixed in the body* and meeting at the point $x'y'$: moreover, let the axis of ξ make the angle ψ with the axis of x; then

$$x = x' + \xi\cos\psi - \eta\sin\psi, \quad y = y' + \xi\sin\psi + \eta\cos\psi,$$

$$\left.\begin{aligned}\dot{x} &= \dot{x}' - (\xi\sin\psi + \eta\cos\psi)\,\omega \\ \dot{y} &= \dot{y}' + (\xi\cos\psi - \eta\sin\psi)\,\omega\end{aligned}\right\}. \tag{2}$$

Or,

$$\dot{x} = \dot{x}' - (y - y')\,\omega, \quad \dot{y} = \dot{y}' + (x - x')\,\omega. \tag{3}$$

These equations show that the velocity of the point xy is made up of two parts—one a velocity of translation, the other a velocity of rotation, as in (1), round an axis through $x'y'$.

For any other definite point, $x''y''$ of the figure we have, in like manner,

$$\dot{x} = \dot{x}'' - (y - y'')\,\omega'', \quad \dot{y} = \dot{y}'' + (x - x'')\,\omega''.$$

Equating these values of $\dot{x}$ and $\dot{y}$ to the former, and comparing the results with the equations

$$\dot{x}' = \dot{x}'' - (y' - y'')\,\omega'', \quad \dot{y}' = \dot{y}'' + (x' - x'')\,\omega'',$$

we see that $\omega'' = \omega$, or the velocity of rotation to be attributed to the body, is the same whatever be the point through which the axis of rotation is supposed to pass.

223. Instantaneous Centre, Body Centrode, Space Centrode.—If we put $\dot{x} = 0$, $\dot{y} = 0$ in equations (2), we get the coordinates of the *instantaneous centre of rotation*, referred to axes fixed in the body. In like manner equations (3) give the coordinates of the same point referred to axes fixed in space. If we call the coordinates of the instantaneous centre ξ_0, η_0; x_0, y_0, respectively, we have

$$\xi_0 = \frac{1}{\omega}(\dot{x}' \sin\psi - \dot{y}' \cos\psi),\ \eta_0 = \frac{1}{\omega}(\dot{x}' \cos\psi + \dot{y}' \sin\psi), \quad (4)$$

$$x_0 = x' - \frac{1}{\omega}\dot{y}',\ \ y_0 = y' + \frac{1}{\omega}\dot{x}'. \quad (5)$$

If $\dot{x}'$, $\dot{y}'$, ω, and ψ are known functions of the time t, we can find from equations (4), by eliminating t, the path described in the body by the instantaneous centre.

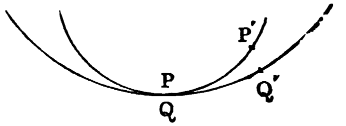
From equations (5) we can find in the same manner the path described by the instantaneous centre in space.

The former of these curves is called the *body centrode*, the latter the *space centrode*.

The student must carefully distinguish between the instantaneous centre and the point of the body which coincides with it at any instant. The latter has no velocity at the instant either in space or in the body; the former (the instantaneous centre) has in general a velocity both in space and in the body.

224. Pure Rolling.—In pure rolling the points of one curve or surface come into contact successively with those of another, the relative tangential velocity of the points of contact being zero. If one curve or surface be fixed in space, the motion of the other consists of a series of rotations round axes through the successive points of contact (*Differential Calculus*, Art. 295). In the case of one plane curve rolling on another, this appears as follows:—

Let QQ' be the curve fixed in space, and PP' the one which rolls on it, P, P' being two consecutive points on the latter. By hypothesis, P has no velocity along the tangent at Q, and at the end of an infinitely short time P' coincides with Q', and the distance between P and Q is then an infinitely small quantity of the second order. Hence, while other points of the body have received infinitely small displacements of the first order, P has received one of the second order, and has, therefore, no velocity in any direction. Hence, during the instant under consideration, the body must be rotating round an axis through P (Art. 217). It is obvious that the acceleration of P in the direction of the tangent at Q is zero; and it can be easily seen that its acceleration in the direction of the normal is in general finite, and equal in magnitude to $U\omega$, where ω is the angular velocity of the body, and U is the velocity of the instantaneous centre of rotation, this point having moved during the instant from Q to Q' in space, and from P to P' in the body.



225. Geometrical Representation of the Motion of a Body moving Parallel to a Fixed Plane.—When a body is moving parallel to a fixed plane, if we can determine the space centrode and the body centrode, the motion of the body is completely determined, as it consists of the rolling, without slipping, of the body centrode on the space centrode.

The geometrical applications of the principles laid down in the present and preceding Articles are numerous and important; but as they do not fall within the scope of the present treatise, the reader is referred for them to Chap. XIX. of the *Differential Calculus*, and to Minchin's *Uniplanar Kinematics*, Chap. III.

226. Velocity of any Given Point of a Body.—In Kinetics the motion of a body is usually regarded as made up of a motion of translation v, and a motion of rotation ω, round an axis through the centre of inertia G. It is sometimes important to determine the velocity of a given point A of the

body. In the case of motion parallel to a fixed plane this is readily done analytically by equations (3).

Otherwise, geometrically:—let p be the distance from A to the axis of rotation through G, then, owing to the rotation, A has a velocity $p\omega$ perpendicular to the plane passing through A and the axis of rotation, and this, combined with the velocity of translation v, gives the velocity of A.

Examples.

1. Show directly that if a body have two equal and opposite velocities of rotation round two parallel axes, the velocity of *any* point is at right angles to the plane containing the parallel axes, and is equal to the distance between the axes multiplied by the angular velocity.

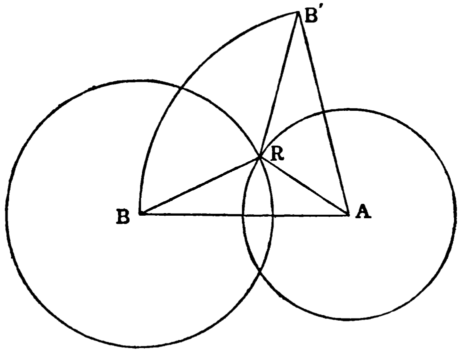
Draw a plane through the point at right angles to the two parallel axes. Describe round the axes circles passing through the point. The component velocities of the point are perpendicular and proportional to the radii of these circles, and the resultant velocity is, therefore, in the direction of the common chord, and proportional to the line joining the centres.

2. Prove that a velocity of rotation round any axis is equivalent to an equal velocity of rotation ω round a parallel axis, together with a velocity of translation ωa along a line at right angles to the plane containing the axes, the distance between which is a.

3. A body receives, in a given order, finite rotations round two parallel axes fixed in space. Determine the magnitude of the equivalent rotation, and the position of its axis.

4. If the parallel axes round which the body receives successive rotations be fixed not in space but in the body, determine the single rotation which would bring the body into the same position.

If A, B are the intersections of the axes *fixed in space*, with a plane at right



angles, R that of the resultant axis, and α, β, χ, the magnitudes of the rotations

round them, then $BAR = -\frac{1}{2}\alpha$, $ABR = \frac{1}{2}\beta$, and the resultant rotation $\chi = \alpha + \beta$, or, $(\alpha - \beta)$, according as α and β are in the same or opposite directions. In the latter case its direction is the same as the greater of the two. If A and B' are the positions of the axes fixed in the body, $B'AR = \frac{1}{2}\alpha$, $AB'R = -\frac{1}{2}\beta$.

5. Two equal and opposite finite rotations round parallel axes bring a body into the same position as a single motion of translation. Determine the direction and magnitude of this translation.

The direction of translation is at right angles to a line which makes with AB or AB' an angle equal to $-\frac{1}{2}\alpha$, or $\frac{1}{2}\alpha$, and the magnitude of the translation $= 2AB \sin\frac{1}{2}\alpha$, or, $2AB' \sin\frac{1}{2}\alpha$.

6. If the direction of the motion of each point of a body be always parallel to a fixed plane, the motion is equivalent to a succession of rotations round the generating lines of a cylinder fixed in space, which is at right angles to the fixed plane.

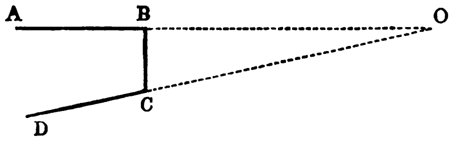
7. A plane area receives a motion of translation in its own plane whose components, parallel to the axes, are a and b; and a rotation θ round the point in the body which, at the beginning of the motion, coincides with the fixed origin. Determine the coordinates of the point, a rotation round which would bring the body into the same position.

$$\textit{Ans.}\ x = \frac{a \sin\frac{1}{2}\theta - b \cos\frac{1}{2}\theta}{2 \sin\frac{1}{2}\theta}, \quad y = \frac{b \sin\frac{1}{2}\theta + a \cos\frac{1}{2}\theta}{2 \sin\frac{1}{2}\theta}.$$

8. Show from these expressions that the amplitude of the rotation is the same as before.

If ϕ be the amplitude, $\sin\frac{1}{2}\phi = \frac{1}{2}\sqrt{\dfrac{a^2 + b^2}{x^2 + y^2}} = \sin^2\frac{1}{2}\theta$; $\therefore\ \phi = \theta$.

9. $ABCD$ is a frame composed of three bars connected by joints at B and C. It is capable of moving in one plane, the points A and D being fixed. Determine the relation between the angular velocities of AB and DC.



At any instant B is moving in a circle round A, or at right angles to AB; and C at right angles to DC. Hence the instantaneous centre of rotation of BC is O, the point of intersection of AB and CD; wherefore $AB \cdot \omega_1 = OB \cdot \omega$, and $DC \cdot \omega_2 = OC \cdot \omega$; hence $\dfrac{\omega_1}{\omega_2} = \dfrac{BO}{AB} \cdot \dfrac{DC}{CO}$ (THOMSON AND TAIT).

10. A bar AB moves in one plane with given angular velocity round A, while at B it is freely jointed to another bar BC, whose extremity C is constrained to move along a fixed straight groove passing through A; find the velocity of C in any position.

Draw a perpendicular to AC at C, and let it meet AB in O; then O is the instantaneous centre of rotation of BC. If v be the velocity of C, and ω the angular velocity of AB, $v = AB \,.\, \frac{OC}{OB} \,.\, \omega = AP \,.\, \omega$, where AP is drawn at right angles to AC to meet BC in P. For the further discussion of this question the reader is referred to Minchin, *Uniplanar Kinematics*, p. 47, or Goodeve, *Elements of Mechanism*, Chap. I. The arrangement of machinery mentioned in this example is called the crank and connecting rod.

11. A bar moves in a horizontal plane with uniform angular velocity round one extremity. To the other extremity a horizontal circle is attached. If the circle be regarded as rotating round its centre, what additional motion must it be considered to have?

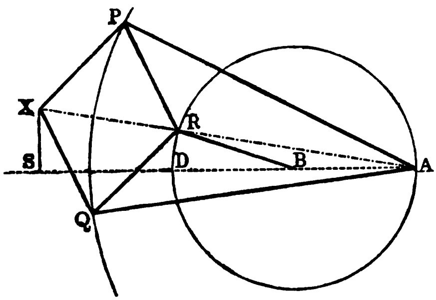
A velocity of translation at right angles to the bar, and equal to $a\omega$, where a is the distance of the centre of the circle from the fixed end of the bar, and ω the angular velocity.

12. If two definite points of a plane figure are constrained to move along two straight lines in its plane, which are fixed in space, the space centrode and the body centrode are circles, the former being double the latter (*Differential Calculus*, Art. 295).

13. In Peaucellier's arrangement find the relation between the velocity of the point describing the straight line and that of one of the adjacent corners of the parallelogram.

M. Peaucellier, in 1864, first succeeded in transforming circular into rectilinear motion by the following arrangement:—A and B are fixed points; AP and AQ are two equal bars which can turn freely round A; BR is another bar turning freely round B, and equal in length to AB; $QRPX$ is a jointed parallelogram composed of four equal bars turning freely round their points of intersection. If a motion be imparted to the system, the points P, Q, R describe circles. That the point X describes a straight line may be shown as follows:—

In any position of the system, since $\angle PRX = \angle QRX$, and $\angle PRA = \angle QRA$, XR and RA are in one straight line; then XPR being an isosceles triangle, and

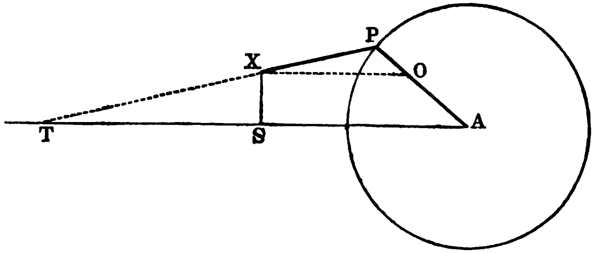


PA a line drawn from the vertex to the base, $AR \,.\, AX = AP^2 - RP^2 =$ const.; wherefore X describes a curve which is the inverse, with respect to A as origin,

of that described by R. Now the point R describes a circle which passes through A; hence X describes a straight line, perpendicular to AB at the point S, where

$$AS \,.\, AD = AP^2 - RP^2.$$

We proceed to find the relation between the velocities of P and X. Draw XO at right angles to SX; then O is the instantaneous centre of rotation of the bar PX.



Let $AP = a$, $PX = b$, BR (in former figure) $= c$; then ω being the angular velocity of AP, ω' that of PX, and v the velocity of X; we have, since O is the instantaneous centre,

$$v = OX \,.\, \omega', \text{ and } OP \,.\, \omega' = AP \,.\, \omega;$$

therefore

$$v = \frac{OX}{OP} \,.\, AP \,.\, \omega = AT \,.\, \omega.$$

Again, if

$$PAT = \theta,\ PTA = \phi, \text{ we have } AT = a \sin\theta\,(\cot\theta + \cot\phi);$$

therefore

$$v = a \sin\theta\,(\cot\theta + \cot\phi)\,\omega,$$

where ϕ is given by the equation

$$a\cos\theta + b\cos\phi = \frac{a^2 - b^2}{2c}.$$

14. A plane area is moving in its own plane; determine the accelerations of any point in it parallel to the tangent and the normal to the space centrode at the instantaneous centre of rotation.

Let x_0, y_0 be the coordinates of a point fixed in the lamina, ξ, η those of any point in it referred to x_0, y_0 as origin, and to axes parallel to those of x, y; then

$$\frac{d\xi}{dt} = -\,\eta\omega, \quad \frac{d\eta}{dt} = \xi\omega,$$

ω being the angular velocity of the body; whence

$$\frac{dx}{dt} = \frac{dx_0}{dt} - \eta\omega,$$

$$\frac{dy}{dt} = \frac{dy_0}{dt} + \xi\omega,$$

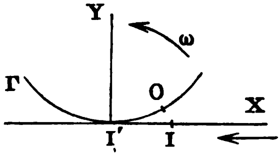
$$\frac{d^2x}{dt^2} = \frac{d^2x_0}{dt^2} - \omega^2\xi - \frac{d\omega}{dt}\eta,$$

$$\frac{d^2y}{dt^2} = \frac{d^2y_0}{dt^2} + \frac{d\omega}{dt}\xi - \omega^2\eta.$$

Call the centrode fixed in space C, that fixed in the body, Γ. The velocity of the point O of the body which coincides at any instant with the instantaneous centre of rotation is zero. At the next instant the instantaneous centre of rotation has moved to the consecutive position on each of the curves C and Γ. At the end of this instant O has a velocity in the normal to C equal to $I'I\omega$, where I, I' are consecutive positions of the instantaneous centre on the tangent to C. Hence the acceleration of O along the tangent to C is zero, and along the normal to C is $\omega^2\frac{d\sigma}{d\theta}$, if we put $I'I = d\sigma$, and $\omega = \frac{d\theta}{dt}$. Now if ρ and ρ' be the radii of curvature of C and Γ, and, if we put $\frac{1}{\rho'} - \frac{1}{\rho} = \frac{1}{R}$, it is easily seen that $\frac{d\theta}{d\sigma} = \frac{1}{R}$. Hence, if x_0y_0 coincide with O, and we take as axes the tangent and normal to C, we have

$$\frac{d^2x}{dt^2} = -\omega^2\xi - \frac{d\omega}{dt}\eta,$$

$$\frac{d^2y}{dt^2} = \omega^2R + \frac{d\omega}{dt}\xi - \omega^2\eta.$$



15. Determine the points of the body which have at any instant (1) no acceleration parallel to the tangent to C at the instantaneous centre of rotation; (2) no acceleration parallel to the normal.

These points consist of two straight lines in the body at right angles to each other, the first of which passes through the instantaneous centre of rotation.

16. Determine at any instant the position of the point in the body having no acceleration.

It is the intersection of the two lines mentioned in the last example.

If α be the angle which the line of non-tangential acceleration (Ex. 15) makes with the axis of x, the coordinates of this point may be expressed in the form

$$\xi = R\sin\alpha\cos\alpha, \quad \eta = R\sin^2\alpha.$$

These expressions readily follow from the equations of Ex. 14. This point is called the acceleration-centre.

17. The acceleration of any point of the body is the same as if the body were turning round the acceleration-centre as an *absolutely fixed point*.

18. All points of the body which have a common acceleration lie on a circle having the acceleration-centre as centre.

19. Find the points of the body for which the acceleration normal to the path described by the point is zero.

Take the centre of rotation as origin of $\xi\eta$; any point is describing a circle round it; hence the line joining the origin to $\xi\eta$ is the normal to the path of

the latter; and if N be the normal acceleration, and r the distance from the instantaneous centre of rotation,

$$N = \frac{\xi}{r}\left(-\omega^2\xi - \frac{d\omega}{dt}\eta\right) + \frac{\eta}{r}\left(\omega^2 R + \frac{d\omega}{dt}\xi - \omega^2\eta\right) = -\omega^2\frac{\xi^2+\eta^2}{r} + \frac{\eta}{r}\omega^2 R.$$

Hence, at any instant, the points for which $N = 0$ lie on the circle

$$\xi^2 + \eta^2 = R\eta.$$

This circle passes through the instantaneous centre of rotation, touches the curve C, and has a radius $= \frac{1}{2}R$. For the reason stated in Ex. 21 it is called the circle of inflexions.—*Differential Calculus*, Art. 290.

20. Determine the points of the system for which the acceleration along the path is zero.

They lie on a circle whose equation, referred to the centre of rotation as origin, is

$$\frac{d\omega}{dt}(\xi^2 + \eta^2) + \omega^2 R\xi = 0,$$

and which passes through the instantaneous centre of rotation and cuts the curve C orthogonally.

The theorems of the last two examples are due to Bresse (*Journal de l'école polytechnique*, t. xx.).

21. Determine at any instant the points of the body which are passing over points of inflexion on their respective paths.

They are the points having no normal acceleration (Ex. 19); for, as $\frac{v^2}{\rho}$ is then zero, and v not zero, ρ must be infinite.

22. Determine the coordinates of the acceleration centre referred—(1) to axes fixed in space; (2) to axes fixed in the body (*see* Article 223).

Let x_1, y_1, ξ_1, η_1 be the coordinates in question, then, x', y' being the space-coordinates of the point of intersection of the body-axes, we have

$$\{\dot\omega^2 + \omega^4\}(x_1 - x') = -\dot\omega\ddot y' + \omega^2\ddot x',$$

$$\{\dot\omega^2 + \omega^4\}(y_1 - y') = \dot\omega\ddot x' + \omega^2\ddot y',$$

$$\{\dot\omega^2 + \omega^4\}\xi_1 = \dot\omega(\ddot x'\sin\psi - \ddot y'\cos\psi) + \omega^2(\ddot x'\cos\psi + \ddot y'\sin\psi),$$

$$\{\dot\omega^2 + \omega^4\}\eta_1 = \dot\omega(\ddot x'\cos\psi + \ddot y'\sin\psi) - \omega^2(\ddot x'\sin\psi - \ddot y'\cos\psi).$$

Section II.—*Kinetics.—Constrained Motion.*

227. **Special Cases of Motion. Degrees of Freedom.**—In order to transform the general equations of motion in such a way as to be of use in particular problems, it is necessary to know something of the special conditions of the problem which it is required to solve.

We have seen in Article 214 that six conditions are required to fix the position of a rigid body, and we have found

accordingly six equations of motion for a body perfectly free. Such a body is said to have *six degrees of freedom* (Art. 215). We have obtained the equations for this case in their most general form (Art. 204), but we shall now adopt the reverse method of procedure, and consider the special equations to be employed for a body having one degree of freedom.

228. **One Degree of Freedom.**—A body is said to have one degree of freedom when its position is limited in such a way as to depend on a single indeterminate quantity. It will be shown subsequently that the variations of the coordinates of any point of a body entirely free are linear functions of six undetermined quantities. If these six quantities are connected together in such a way that one being given all the rest are determined, the body has one degree of freedom.

The simplest cases of one degree of freedom occur when some of the six undetermined displacements are zero. We shall consider here only two cases.

(1). If the motion of the body be limited to a series of pure translations, and the path of one of its points be assigned.

(2) If the motion of the body be limited to a rotation round an axis fixed in space.

In the first case the problem is readily reducible to that of the constrained motion of a particle.

This reduction is most easily effected by employing D'Alembert's Principle as expressed by Lagrange. In fact we have

$$\Sigma\left\{\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\frac{d^2z}{dt^2}\right)\delta z\right\} = 0.$$

Now, by the conditions of the question, δx, δy, δz must be the same for every point of the body, and ds being the arc of the curve described by the centre of inertia,

$$\delta x = \frac{d\bar{x}}{ds}\delta s, \quad \delta y = \frac{d\bar{y}}{ds}\delta s, \quad \delta z = \frac{d\bar{z}}{ds}\delta s.$$

Making these substitutions, we obtain the single equation of motion,

$$\left(\Sigma m \frac{d^2x}{dt^2}\right)\frac{d\bar{x}}{ds} + \left(\Sigma m \frac{d^2y}{dt^2}\right)\frac{d\bar{y}}{ds} + \left(\Sigma m \frac{d^2z}{dt^2}\right)\frac{d\bar{z}}{ds}$$

$$= (\Sigma X)\frac{d\bar{x}}{ds} + (\Sigma Y)\frac{d\bar{y}}{ds} + (\Sigma Z)\frac{d\bar{z}}{ds};$$

or, as $$ds^2 = d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2,$$

we have finally, if we put $\mathfrak{M}$ for the whole mass of the body,

$$\mathfrak{M}\frac{d^2s}{dt^2} = S, \qquad (1)$$

where S is the sum of the components of all the applied forces along the tangent to the path of the centre of inertia; but this is obviously the equation required for determining the constrained motion of a particle.

229. **Motion of a Body round an Axis fixed in Space.**—The condition of equilibrium of a rigid body having a fixed axis is, that the moment of the forces round this axis should be zero. Take the fixed axis as axis of x, then the single equation of motion is the first of equations (18) or (16), Art. 204, according as the forces acting on the body are impulsive or continuous. Adopting the notation of Art. 210, the equation of motion is thus:

$$H_1 - H_1' = \dot{L}, \text{ or } \frac{dH_1}{dt} = L.$$

Let p be the perpendicular on the axis from any point P of the body, ω its angular velocity at any instant, and I its moment of inertia round the axis; then, since $p\omega$ is the velocity of the particle P, its moment of momentum is $mp^2\omega$, and $H_1 = \omega\Sigma mp^2 = I\omega$. Substituting this value for H_1, and remembering that I is constant, we obtain as the equation of motion in the case of impulses

$$I(\omega - \omega') = \dot{L}, \qquad (2)$$

and in the case of continuous forces

$$I\frac{d\omega}{dt} = L. \tag{3}$$

Equation (3) was obtained before in Art. 138 by a different method.

230. **Equation of Vis Viva for a Body moving round a Fixed Axis.**.—The expression for the vis viva of a body moving round a fixed axis has been given already, Art. 133. If we take the fixed axis for the axis of x, we have, as the equation of vis viva,

$$I\omega^2 = 2\Sigma\int(Ydy + Zdz) + c. \tag{4}$$

Examples.

1. To the ends of a thin light piece of wood are fastened spheres of lead whose masses are P and P'. The piece of wood turns on a horizontal axis through its middle point. Its length being $2l$, and its mass negligible, determine the time of a small oscillation, the spheres being so small that the squares of their radii are negligible as compared with l.

Ans. $\pi\sqrt{\frac{l}{g}}\cdot\sqrt{\frac{P+P'}{P-P'}}$.

By changing P, and comparing the times of oscillation, an apparatus of the kind mentioned can be used to verify the Laws of Motion.

2. A heavy pendulum, capable of revolving round a horizontal axis, is struck when at rest by a bullet moving in a horizontal direction at right angles to the fixed axis. The bullet remains in the pendulum. If b be the distance of the extremity of the pendulum from the axis, c the distance traversed by that extremity under the influence of the shot, a the distance from the axis at which the bullet penetrates, v the velocity of the bullet at impact, m its mass, M that of the pendulum, k its radius of gyration round the fixed axis, and p the distance of the latter from the centre of inertia; prove that

$$v = \frac{c}{mab}\sqrt{\{g(Mk^2 + ma^2)(Mp + ma)\}}.$$

A pendulum such as that described above is called a Ballistic Pendulum. has been employed by numerous Physicists to determine the velocity of bullets.

3. A plane area is made to rotate with an angular velocity ω' round a fixed axis in its own plane by the expenditure of a given amount of work. When rotating it strikes a sphere of mass m, at a distance a from the fixed axis, whose

velocity at the instant of impact is zero. Determine the moment of inertia of the plane area round the fixed axis in order that the velocity imparted to the sphere should be a maximum.

If R be the impulse on the sphere in the first period of impact, v its velocity, and ω the angular velocity of the lamina at the end of this period,

$$mv = R, \quad I(\omega - \omega') = -aR, \quad a\omega = v,$$

whence
$$R = \frac{maI\omega'}{I + ma^2}.$$

The whole impulse given to the sphere is $(1 + e)R$. Hence R is to be a maximum; but $I\omega'^2 =$ given constant; therefore $\frac{\sqrt{I}}{I + ma^2} =$ maximum; and therefore $I = ma^2$.

4. In Atwood's machine, if the pulley be not perfectly rough, and slipping takes place, determine the motion: the weight of the rope and the friction of the pulley on the axle being neglected.

If an acceleration equal and opposite to that by which it is actually animated were applied to each element of the string it would be in equilibrium; but the mass of the string being negligible, the force corresponding to this acceleration is zero *q.p.* Hence the other forces acting on the element of the string are in equilibrium, and μ being the coefficient of friction, and T, T' the tensions of the two ends of the rope (Minchin, *Statics*), $T' = Te^{-\mu\pi} = \lambda T$.

If z be the height from the ground of the ascending weight W', M the mass of the pulley, k its radius of gyration, a its radius, θ the angle through which it has turned, we have also

$$\frac{T' - W'}{W'} g = \frac{d^2z}{dt^2} = \frac{W - T}{W} g,$$

$$Mk^2 \frac{d^2\theta}{dt^2} = a(T - T').$$

If the pulley be homogeneous, $k^2 = \frac{a^2}{2}$, and we have finally,

$$T = \frac{2WW'}{\lambda W + W'}, \quad \frac{d^2z}{dt^2} = \frac{\lambda W - W'}{\lambda W + W'} g,$$

$$a\frac{d^2\theta}{dt^2} = 4(1 - \lambda) \frac{WW'}{M(\lambda W + W')}.$$

5. Taking into account the friction on the axle, and supposing the outside of the pulley to be perfectly rough, and the inside to slip on the axle, determine the motion.

The mass of the string being neglected, we may, as in the last example, regard it as acted on by a system of forces in equilibrium. Hence (as this equilibrium would not be disturbed if the string were rigid) the tensions T and T' at its extremities must equilibrate the pressure and friction exerted by the pulley against the string; and, conversely, T and T' must be equivalent to the

pressure and friction exerted by the string against the pulley. Hence we may consider the pulley as acted on by the forces T, T', and its own weight; and if P be the horizontal, and Q the vertical, pressure on the axle, and μ the coefficient of friction, since the centre of inertia of the pulley is at rest, we have (Art. 206), $P = \mu Q$, $Q = T + T' + Mg - \mu P$. The moment of the couple resulting from the friction is $\mu(P + Q)a$, where a is the radius of the axle, and may therefore be written in the form $\beta(T + T' + Mg)$, where $(1 + \mu^2)\beta = \mu(1 + \mu)a$.

Substituting for the equation $Mk^2 \dfrac{d^2\theta}{dt^2} = a(T - T')$ of Ex. 4,

the equation $$Mk^2 \frac{d^2\theta}{dt^2} = a(T - T') - \beta(T + T' + Mg);$$

nd remembering that as the pulley is perfectly rough, $a\dfrac{d\theta}{dt} = \dfrac{dz}{dt}$, we obtain, if we put $\nu = \dfrac{\beta}{a}$ and assume that $k^2 = \dfrac{a^2}{2}$,

$$T = \frac{(1 + 2\nu)Mg + 4(1 + \nu)W'}{Mg + 2(1 - \nu)W + 2(1 + \nu)W'} \cdot W,$$

$$T' = \frac{(1 - 2\nu)Mg + 4(1 - \nu)W}{Mg + 2(1 - \nu)W + 2(1 + \nu)W'} \cdot W',$$

$$\frac{d^2z}{dt^2} = \frac{(1 - \nu)W - (1 + \nu)W' - \nu Mg}{(1 - \nu)W + (1 + \nu)W' + \frac{1}{2}Mg} \cdot g.$$

6. If the pulley be not perfectly rough, and slipping of the string on the pulley takes place, determine the motion, taking into account the friction on the axle, and supposing the inside of the pulley to slip as before.

In this case, as in Ex. 4, the acceleration of the weights is quite independent of the mass and size of the pulley, and we have

$$T = \frac{2WW'}{\lambda W + W'}, \quad T' = \lambda T, \quad \frac{d^2z}{dt^2} = \frac{\lambda W - W'}{\lambda W + W'} g.$$

$$a \frac{d^2\theta}{dt^2} = \left\{ \frac{4\{1 - \nu - \lambda(1 + \nu)\} WW'}{Mg(\lambda W + W')} - 2\nu \right\} g.$$

231. Moments of Momentum of Body having fixed Axis.—The expression for the moment of momentum of a rigid body round an axis fixed in space was found in Art. 229. Adopting the notation of that article, we shall now, by a more general method, obtain expressions for the moments of momentum round each of the three coordinate axes.

We have (Art. 222), since the body is supposed to be rotating round the axis of x,

$$\dot{x} = o, \quad \dot{y} = -z\omega, \quad \dot{z} = y\omega;$$

whence by (22), Art. 210,

$$H_1 = \omega\Sigma m(y^2 + z^2), \quad H_2 = -\omega\Sigma mxy, \quad H_3 = -\omega\Sigma mxz. \quad (5)$$

Also, by differentiation, and substitution of their values for $\dot{x}$, $\dot{y}$, and $\dot{z}$, we obtain

$$\left.\begin{aligned} \frac{dH_1}{dt} &= \frac{d\omega}{dt}\Sigma m(y^2 + z^2), \\ \frac{dH_2}{dt} &= -\frac{d\omega}{dt}\Sigma mxy + \omega^2\Sigma mxz, \\ \frac{dH_3}{dt} &= -\frac{d\omega}{dt}\Sigma mxz - \omega^2\Sigma mxy \end{aligned}\right\}. \quad (6)$$

If the axis of rotation be a principal axis for the origin, equations (5) and (6) become;

$$H_1 = A\omega, \; H_2 = 0, \; H_3 = 0, \frac{dH_1}{dt} = A\frac{d\omega}{dt}, \; \frac{dH_2}{dt} = 0, \; \frac{dH_3}{dt} = 0, \quad (7)$$

where A is the moment of inertia of the body round the fixed axis.

232. **Acceleration of any Point of a Body having a Fixed Axis.**—If we differentiate the expression for $\dot{x}$, $\dot{y}$, and $\dot{z}$ given in Art. 231, and then substitute in the results thus obtained the values of $\dot{x}$, $\dot{y}$, $\dot{z}$ already employed, we get

$$\ddot{x} = 0, \quad \ddot{y} = -\dot{\omega}z - \omega^2 y, \quad \ddot{z} = \dot{\omega}y - \omega^2 z. \quad (8)$$

233. **Stresses on the Axis of Rotation.**—We have seen that D'Alembert's Principle furnishes at once the single equation of motion which is required to determine the velocity and position of a body rotating round a fixed axis. The same principle enables us to write down the equations which are required to determine the stresses on the axis.

In order to determine these stresses, we may regard the body as compelled to rotate round the fixed axis by forces acting on the body at any two points on the axis. The body is then to be considered free, but the magnitude of the forces replacing the constraints is such as to compel the body to rotate round the given axis.

These forces are obviously equivalent to a single force passing through the origin, and a couple whose axis is perpendicular to the fixed axis.

The stresses on the axis are equal and opposite to the forces by which we have supposed the axis to be replaced.

234. **Stresses due to Impulses.**—In this case we shall suppose the stresses to consist of an impulse passing through the origin whose components are $\underset{\cdot}{X}_0$, $\underset{\cdot}{Y}_0$, $\underset{\cdot}{Z}_0$, together with an impulsive couple whose components round the axes of y and z are $\underset{\cdot}{M}_0$ and $\underset{\cdot}{N}_0$.

If we suppose the body to start from rest, equations (17) and (18), Art. 204, become, by Art. 231,

$$\left.\begin{aligned} \Sigma\underset{\cdot}{X} - \underset{\cdot}{X}_0 &= \Sigma m\dot{x} = 0, \\ \Sigma\underset{\cdot}{Y} - \underset{\cdot}{Y}_0 &= \Sigma m\dot{y} = -\omega\Sigma mz = -\omega\mathfrak{M}\bar{z}, \\ \Sigma\underset{\cdot}{Z} - \underset{\cdot}{Z}_0 &= \Sigma m\dot{z} = \omega\Sigma my = \omega\mathfrak{M}\bar{y} \end{aligned}\right\}, \qquad (9)$$

$$\left.\begin{aligned} \underset{\cdot}{L} &= \omega\Sigma m(y^2 + z^2) = I\omega, \\ \underset{\cdot}{M} - \underset{\cdot}{M}_0 &= -\omega\Sigma mxy, \\ \underset{\cdot}{N} - \underset{\cdot}{N}_0 &= -\omega\Sigma mxz \end{aligned}\right\}. \qquad (10)$$

When ω has been found from the first of equations (10) the remaining five equations determine the stresses.

If the fixed axis be a principal axis at the origin the last two equations become

$$\underset{\cdot}{M} - \underset{\cdot}{M}_0 = 0, \quad \underset{\cdot}{N} - \underset{\cdot}{N}_0 = 0.$$

Hence, if a body having a fixed axis, which is a principal axis for one of its points, be set in motion by an impulsive couple whose plane is perpendicular to the axis, there is no impulsive stress couple.

From this we infer, that if a body, having a fixed point O, be acted on by an impulsive couple in one of the principal planes at O, it will commence to turn round the axis perpendicular to the plane of the impulsive couple. Again, if the body be acted on by an impulse whose line of direction is situated in one of the principal planes at O, it will commence to turn round the normal to this plane.

For a *free* body, likewise, having O for its centre of inertia, these results are true; but, in the case of the second, the body has also an initial motion of translation.

If the body, before the action of the impulses $\underset{\cdot}{X}$, &c., be already rotating round the fixed axis with an angular velocity ω', equations (9) and (10) still hold good in their final form, provided $\omega - \omega'$ be substituted for ω.

If we suppose the origin O to be the centre of suspension, or point in which the fixed axis is met by the perpendicular p from the centre of inertia G, and take the axis of y to coincide with this line, and if we denote the sum of the components of the applied impulses parallel and perpendicular to OG by $\underset{\cdot}{P}$ and $\underset{\cdot}{Q}$, and the corresponding impulsive stresses by $\underset{\cdot}{P}_0$ and $\underset{\cdot}{Q}_0$, equations (9) become

$$\Sigma\underset{\cdot}{X} - \underset{\cdot}{X}_0 = 0, \quad \underset{\cdot}{P} - \underset{\cdot}{P}_0 = 0, \quad \underset{\cdot}{Q} - \underset{\cdot}{Q}_0 = \mathfrak{M}p\omega. \quad (11)$$

235. **Centre of Percussion.**—If a body receive a blow which makes it begin to rotate round a fixed axis, without causing any impulsive pressure on the axis, the point in which the direction of the blow intersects the plane containing the fixed axis and the centre of inertia is called the centre of percussion. In order that such a point should exist, both the axis and the line of direction of the impulse must fulfil certain conditions, which we proceed to investigate.

In this case, the fixed axis being, as before, the axis of x, we have, by hypothesis, $\underset{\cdot}{X}_0 = 0$, $\underset{\cdot}{Y}_0 = 0$, $\underset{\cdot}{Z}_0 = 0$, $\underset{\cdot}{M}_0 = 0$, $\underset{\cdot}{N}_0 = 0$. If we denote the components of the impulse due to the blow by $\underset{\cdot}{X}$, $\underset{\cdot}{P}$, $\underset{\cdot}{Q}$; and the components of the impulsive couple which it produces by $\underset{\cdot}{L}$, $\underset{\cdot}{M}$, $\underset{\cdot}{N}$; equations (11) and (10) become

$$\left.\begin{aligned} &\underset{\cdot}{X} = 0, \quad \underset{\cdot}{P} = 0, \quad \underset{\cdot}{Q} = \mathfrak{M}p\omega, \\ &\underset{\cdot}{L} = I\omega, \; \underset{\cdot}{M} = -\omega\Sigma mxy, \; \underset{\cdot}{N} = -\omega\Sigma mxz \end{aligned}\right\}. \quad (12)$$

Since $\dot{X} = 0$, and $\dot{P} = 0$, *the centre of inertia must lie in a plane through the fixed axis, at right angles to the direction of the impulse.*

Again, since $\dot{X} = 0$, the direction of the blow may be supposed to lie in the plane of yz, and therefore the resulting couple has no components in the plane of zx or of xy; accordingly, $\dot{M} = 0$ and $\dot{N} = 0$. Hence, we have $\Sigma mxy = 0$, and $\Sigma mxz = 0$; consequently, *the axis of rotation must be a principal axis for the point in which it is met by its shortest distance from the line of direction of the impulse.* If, now, q be the distance from the fixed axis of the line of action of the blow, $\dot{L} = Qq$, and therefore $\mathfrak{M}pq = I$.

If $\mathfrak{M}k^2$ be the moment of inertia of the body round an axis through its centre of inertia parallel to the fixed axis, $I = \mathfrak{M}(k^2 + p^2)$. (*Integral Calculus*, Chap. X.)

Hence $$q = \frac{k^2 + p^2}{p}.$$

Accordingly, *the distance of the centre of percussion from the fixed axis is the same as that of the centre of oscillation.* (Art. 136.)

Moreover, if ξ, η, ζ be the coordinates of any point relatively to the centre of inertia,

$$\int xz\,dm = \mathfrak{M}\bar{x}\bar{z} + \int \xi\zeta\,dm;$$

hence, if the axis of suspension be parallel to a principal axis through the centre of inertia, $\bar{x} = 0$, *and the shortest distance between the direction of the blow and the fixed axis passes through the centre of inertia, and the centre of percussion coincides with the centre of oscillation.*

236. **Stress on Fixed Axis during Rotation.**—In accordance with Art. 233, and following the analogy of Art. 234, we shall suppose the stress at any instant to consist of a force passing through the origin, whose components are X_0, Y_0, and Z_0, together with a couple whose components round the axes of y and z are M_0 and N_0.

In this case, by Arts. 231 and 232, equations (15) and (16), Art. 204, become

$$\left.\begin{aligned} \Sigma X - X_0 &= 0, \\ \Sigma Y - Y_0 &= -\mathfrak{M}\bar{z}\dot{\omega} - \mathfrak{M}\bar{y}\omega^2, \\ \Sigma Z - Z_0 &= \mathfrak{M}\bar{y}\dot{\omega} - \mathfrak{M}\bar{z}\omega^2 \end{aligned}\right\}. \qquad (13)$$

$$\left.\begin{aligned} I\dot{\omega} &= L, \\ -\dot{\omega}\Sigma mxy + \omega^2\Sigma mxz &= M - M_0, \\ -\dot{\omega}\Sigma mxz - \omega^2\Sigma mxy &= N - N_0 \end{aligned}\right\}. \qquad (14)$$

If the axis of rotation be a principal axis for the origin, the last two equations reduce to $M - M_0 = 0$, $N - N_0 = 0$.

If also the couple resulting from the applied forces be perpendicular to the axis of rotation, we shall have

$$M_0 = 0, \text{ and } N_0 = 0.$$

Accordingly, in this case, the stress couple vanishes when the axis of rotation is a principal axis for the origin.

If the axis of rotation pass through the centre of inertia of the body, we have

$$\Sigma X - X_0 = 0, \quad \Sigma Y - Y_0 = 0, \quad \Sigma Z - Z_0 = 0.$$

Accordingly, *if a body be rotating round a principal axis through its centre of inertia, no external forces being supposed to act, there is no stress on the axis, and the body will continue to rotate round that axis with a uniform angular velocity.*

This result was obtained before in Article 98.

If we make the same hypotheses as those at the end of Art. 234, and adopt a similar notation, equations (13) become

$$\Sigma X - X_0 = 0, \quad P - P_0 = -\mathfrak{M}p\omega^2, \quad Q - Q_0 = \mathfrak{M}p\dot{\omega}. \quad (15)$$

These equations of motion of the centre of inertia can of course be obtained directly from the consideration that this

point is describing a circle round the origin with an angular velocity ω.

In general, $\dot{\omega}$ and ω can be determined from the first of equations (14), and the stresses can then be found from the remaining equations of this Article.

EXAMPLES.

1. A rigid body is turning round a fixed axis under the influence of a couple, whose axis is parallel to the axis of rotation: what condition must be fulfilled in order that the axis should suffer a pressure at only one point? (Schell, *Theorie der Bewegung und der Kräfte.*)

The axis of rotation must be a principal axis at this point. The pressure is then at right angles to the axis.

2. If the pressure at the fixed point vanishes, what further condition must be fulfilled?

The point must be the centre of inertia.

3. If a homogeneous cubical mass at rest receive an impulse, determine the resulting motion.

4. A body starting from rest turns under the action of gravity round a fixed horizontal axis, which is a principal axis at the centre of suspension. Find the stress on the axis.

Take the centre of suspension (Art. 136) for origin, and the fixed axis for that of x.

Let θ be the angle which the line joining the origin to the centre of inertia makes at any instant with a horizontal line perpendicular to the fixed axis, then $\omega = \frac{d\theta}{dt}$, and the axis of x being a principal axis at the origin, the stress couple is zero. Again, m being the mass of the body, $L = mpg\cos\theta$, and therefore,

$$\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \frac{gp\cos\theta}{k^2+p^2};$$

whence, by integration,

$$\omega^2 = \left(\frac{d\theta}{dt}\right)^2 = \frac{2gp}{k^2+p^2}(\sin\theta - \sin\alpha),$$

where α is the initial value of θ.

Finally, $P = mg\sin\theta$, and $Q = mg\cos\theta$; whence, substituting their values for P, Q, ω^2, and $\dot{\omega}$ in equations (15), we obtain

$$P_0 = mg\left\{\frac{k^2+3p^2}{k^2+p^2}\sin\theta - \frac{2p^2}{k^2+p^2}\sin\alpha\right\},\quad Q_0 = mg\,\frac{k^2}{k^2+p^2}\cos\theta.$$

5. In Ex. 3 find the position of the body in which the stress on the axis is a minimum.

From the expressions for P_0 and Q_0, we obtain

$$P_0^2+Q_0^2=\frac{m^2g^2}{(k^2+p^2)^2}\{k^4+2k^2p^2\sin\theta(3\sin\theta-2\sin\alpha)+p^4(3\sin\theta-2\sin\alpha)^2\},$$

and, since θ is never less than α, this expression is a minimum when $\theta=\alpha$.

6. A bar, revolving with an angular velocity Ω round a fixed axis perpendicular to its length, strikes perpendicularly against a fixed obstacle; find the impulses against the obstacle and the axis, and the angular velocity of the bar, after collision.

Let O be the point in which the fixed axis meets the bar, G its centre of inertia, A the point at which it strikes the obstacle, m its mass, and k its radius of gyration round an axis through G parallel to the fixed axis; let R' and Q' be the magnitudes of the impulses produced by the obstacle and the axis in the first period of impact, R'' and Q'' those produced in the second period, and ω the angular velocity after collision; then, if $OG=a$, $GA=b$, since the velocity of the bar is reduced to zero in the first period, we have

$$R'+Q'=ma\Omega,\quad R'(a+b)=m(k^2+a^2)\Omega;$$

whence,

$$R'=\frac{m(k^2+a^2)\Omega}{a+b};\quad Q'=m\frac{ab-k^2}{a+b}\Omega.$$

Again, since in the second period the bar starts from rest, we have

$$R''+Q''=ma\omega,\quad R''(a+b)=m(k^2+a^2)\omega,$$

and also (Art. 78),

$$R''=eR',\quad \text{whence}\quad Q''=eQ',\quad \omega=e\Omega,$$

since Q'' and ω are the same functions of R'', which Q' and Ω are of R'.

It is to be observed that in the equations above the algebraical signs of the angular velocities have not been taken into account, and that the direction of ω is opposite to that of Ω.

Finally, if R and Q be the total impulses,

$$R=(1+e)R'=(1+e)\frac{m(k^2+a^2)\Omega}{a+b},\quad Q=(1+e)m\frac{ab-k^2}{a+b}\Omega.$$

When $ab=k^2$, the point A is the centre of percussion and $Q=0$. This agrees with the result arrived at in Art. 235.

7. A bar, revolving as in Ex. 6, strikes against a sphere whose centre is moving with a velocity U in a direction perpendicular to the bar; find the magnitudes of the impulses, and the velocities of the bar and sphere, after collision.

Let M be the mass of the sphere, u' and u the velocities of its centre, and ω' and ω the angular velocities of the bar, at the end of the first and of the second period of impact; then, since the impulses tend to diminish both the velocity of the centre of inertia and the angular velocity of the bar, we have

$$R'+Q'=ma(\Omega-\omega'),\quad R'h=m(k^2+a^2)(\Omega-\omega'),\quad R'=M(u'-U),$$

where $h=a+b$.

At the end of the first period of impact the relative velocity of the colliding points is zero, and therefore, $h\omega'=u'$.

Let $\Omega - \omega' = \varpi'$, then we have

$$R' = \frac{m(k^2 + a^2)}{h}\varpi', \quad Q' = \frac{m(ab - k^2)}{h}\varpi',$$

and also

$$m(k^2 + a^2)\varpi' = Mh(u' - U) = Mh(h\omega' - U) = Mh\{h(\Omega - \varpi') - U\},$$

hence $$\{m(k^2 + a^2) + Mh^2\}\varpi' = Mh(h\Omega - U).$$

Again,

$$R'' + Q'' = ma(\omega' - \omega),\ R''h = m(k^2 + a^2)(\omega' - \omega),\ R'' = M(u - u'), \text{ and } R'' = eR'.$$

Hence we have

$$R = (1 + e)R' = (1 + e)\frac{Mm(k^2 + a^2)(h\Omega - U)}{m(k^2 + a^2) + Mh^2},$$

$$Q = (1 + e)Q' = (1 + e)\frac{Mm(ab - k^2)(h\Omega - U)}{m(k^2 + a^2) + Mh^2}.$$

Also,

$$\omega = \Omega - (1 + e)\varpi' = \Omega - (1 + e)\frac{Mh(h\Omega - U)}{m(k^2 + a^2) + Mh^2},$$

$$u = U + \frac{R}{M} = U + (1 + e)\frac{m(k^2 + a^2)(h\Omega - U)}{m(k^2 + a^2) + Mh^2},$$

Here, as in the former Example, $Q = 0$ when the impact takes place at the centre of percussion.

8. Show that the results in Ex. 6 can be deduced immediately from those in Ex. 7.

Make $U = 0$ and $M = \infty$ in Ex. 7.

9. Find at what point of its length the bar should strike the sphere in order that the impulse of the blow should be a maximum.

If we put $m(k^2 + a^2) = I$, we have to determine h, so that

$$\frac{h\Omega - U}{I + Mh^2}$$

shall be a maximum. Hence, to determine h we have the quadratic equation

$$M\Omega h^2 - 2MUh - I\Omega = 0.$$

By assuming

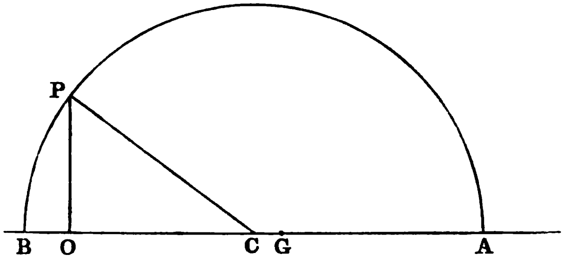
$$U = r\Omega, \quad \text{and } I = Mp^2,$$

this equation becomes

$$h^2 - 2rh - p^2 = 0.$$

We have then the following construction for the two values of h. At O erect OP perpendicular to the bar, and make it equal to p, take OC in the direction of G equal to r, with C as centre, and CP as radius describe a circle; it will meet

the bar in the points required. The value of h, which is greater than r, makes the expression for R a maximum; the other value of h makes this expression a minimum, but at the same time makes R negative. Thus both values of h make R irrespective of sign a maximum; but one impulse is opposite in direction to the other.



If the sphere, when struck, has no velocity in a direction perpendicular to the bar, we have $h = p$ when R is a maximum.

10. Find the point of impact in order that the impulse on the fixed axis should be a maximum.

If we put $(k^2 + a^2) = K^2$, we have to determine h so that

$$\frac{(ah - K^2)(h\Omega - U)}{mK^2 + Mh^2}$$

shall be a maximum. We have then for h the quadratic

$$h^2 - 2\frac{K^2(MU - ma\Omega)}{M(aU + K^2\Omega)}h - \frac{mK^2}{M} = 0.$$

By assuming

$$r'M(aU + K^2\Omega) = K^2(MU - ma\Omega),$$

and (as in last Example),

$$Mp^2 = mK^2,$$

the equation for h becomes

$$h^2 - 2r'h - p^2 = 0,$$

and the two values of h are determined by a construction similar to that of the last Example.

If the fixed axis pass through the centre of inertia we have $a = 0$, and the points for which Q is a maximum coincide with those for which R is a maximum.

If $aU + K^2\Omega = 0$, one value of h is zero, and the percussion on the axis is a maximum when the sphere strikes at the axis.

SECTION III.—*Kinetics—Free Motion Parallel to a Fixed Plane.*

237. **Equations of Motion.**—The motion of a body relative to its centre of inertia consists at any instant of a rotation round some axis through that point. Moreover, in the case here considered, this axis must be at right angles to the fixed plane, and is fixed in space if the centre of inertia be regarded as invariable. Now, by Art. 209, the motion relative to the centre of inertia is the same as if that point were fixed in space, the forces remaining unaltered. Hence, taking the plane of yz for the fixed plane, we have, to determine the motion of the body, the equations

$$\mathfrak{M}\frac{d^2\bar{y}}{dt^2} = \Sigma Y, \quad \mathfrak{M}\frac{d^2\bar{z}}{dt^2} = \Sigma Z, \quad \mathfrak{M}k^2\frac{d^2\theta}{dt^2} = L, \qquad (1)$$

where $\bar{y}$ and $\bar{z}$ are the coordinates of the centre of inertia, k the radius of gyration round an axis through it at right angles to the fixed plane, and L the moment of the applied forces.

If the axis of rotation through the centre of inertia be always parallel to a line fixed in space, it is plain that the last of these equations holds good no matter whether the whole motion of the body be parallel to a fixed plane or not. In the latter case the only difference will be that an additional equation, viz.,

$$\mathfrak{M}\frac{d^2\bar{x}}{dt^2} = \Sigma X,$$

will be required to determine the motion of the centre of inertia. In any case, therefore, the motion of the body is determined, when we know the motion of its centre of inertia, and the angular motion relative to that point.

238. **Connexion of the Angular Velocity with the Velocity of the Centre of Inertia.**—As the motion is parallel to a fixed plane, the parallel section of the body passing through the centre of inertia must at each instant be rotating round a point in its own plane (Art. 219). If ρ be the

distance from this point (the instantaneous centre of rotation) to the centre of inertia, s the path of the latter, and ω the angular velocity, then $\rho\omega = \frac{ds}{dt}$, as is obvious. Also $\omega = \frac{d\theta}{dt}$.

Examples.

1. A body is moving parallel to a fixed plane under the action of forces which are in equilibrium: show that the locus of the instantaneous centre of rotation in the body is a circle, having the centre of inertia for centre, and a radius $\frac{v}{\omega}$, where v is the velocity of the centre of inertia, and ω the angular velocity.

2. The locus of the instantaneous centre of rotation in space, under the circumstances of Ex. 1, is a straight line parallel to the path of the centre of inertia, and at a distance from it equal to $\frac{v}{\omega}$.

3. If a body move parallel to a fixed plane, and be acted on by a constant couple, lying in the plane; show that the locus of the instantaneous centre of rotation in space is an equilateral hyperbola.

4. An inextensible string, whose mass is negligible, passes over the line of intersection of two smooth inclined planes. Each end of the string passes under and round a smooth circular homogeneous cylinder, to which it is attached, and which rests on one of the inclined planes. The line of intersection of the inclined planes is parallel to the axes of the cylinders, and perpendicular to a vertical plane containing their centres of inertia and the string. Determine the tension of the string.

As in Ex. 4, Art. 230, the portion of the string wrapped round one of the cylinders may be regarded as in equilibrium under the action of the tensions at its extremities and of the pressure produced by the cylinder. Hence all the forces exerted by the string on the cylinder are equivalent to the tension T acting at the point of contact of the cylinder with the inclined plane.

If s and s' be the distances at any time of the points of contact of the cylinders and inclined planes, from the point of intersection of the latter with the vertical plane perpendicular to them; θ and θ' the angles through which the cylinders have turned from their initial positions; a and a' their radii; m and m' their masses; and i and i' the inclinations of the inclined planes to the horizon, the equations of motion are

$$m\frac{d^2s}{dt^2} = mg\sin i - T, \quad m\frac{a^2}{2}\frac{d^2\theta}{dt^2} = Ta,$$

$$m'\frac{d^2s'}{dt^2} = m'g\sin i' - T, \quad m'\frac{a'^2}{2}\frac{d^2\theta'}{dt^2} = Ta'.$$

If σ be the distance the string has slipped at any time along the inclined

planes, and b and b' the initial values of s and s', we have, since the string is inextensible,

$s = b + a\theta + \sigma, \quad s' = b' + a'\theta' - \sigma,$ and therefore $s + s' = b + b' + a\theta + a'\theta'.$

Differentiating twice we obtain, by means of the equations of motion,

$$T = \tfrac{1}{3}\frac{mm'}{m+m'}g\,(\sin i + \sin i').$$

The motion can then be completely determined.

239. **Vis Viva.**—It was shown in Art. 134, that the *vis viva* of any system $\Sigma mv^2 = \mathfrak{M}V^2 + \Sigma mv'^2$, where $\mathfrak{M}$ is the entire mass of the system, V the velocity of its centre of inertia, and v' the velocity, relative to the centre of inertia, of any particle m. If the body be moving parallel to a fixed plane, the motion relative to the centre of inertia is a rotation round an axis fixed in the body, whose direction is fixed in space. Hence $\Sigma mv'^2 = \mathfrak{M}k^2\omega^2$ (Art. 133), and the equation of *vis viva* becomes

$$\mathfrak{M}(V^2 + k^2\omega^2) = 2\Sigma\int(Ydy + Zdz) + C. \qquad (2)$$

The equation of *vis viva* may be put into another shape which is sometimes useful. If I be the moment of inertia of the body round the instantaneous axis of the rotation by which the whole motion of the body may be represented, then

$$\Sigma mv^2 = I\omega^2.$$

Again, if y' and z' be the coordinates of any point referred to that space point as origin which coincides with the instantaneous centre of rotation, Art. 238, then

$$\frac{dy'}{dt} = -\omega z', \quad \frac{dz'}{dt} = \omega y';$$

hence the equation of *vis viva* assumes the form

$$\frac{d}{dt}(I\omega^2) = 2\Sigma\left(Y\frac{dy'}{dt} + Z\frac{dz'}{dt}\right) = 2\omega\,\Sigma\,(y'Z - z'Y);$$

therefore
$$\frac{1}{2\omega}\frac{d}{dt}(I\omega^2) = J, \qquad (3)$$

where J is the moment of the applied forces, round the instantaneous axis of rotation.

240. **Moment of the Forces of Inertia.**—If b and c be the coordinates of any point, fixed or movable, the moment of the applied forces, round an axis through it parallel to the axis of x, must be equal and opposite to the moment of the forces of inertia round the same; hence, calling the former moment J, we have

$$\Sigma m \left\{(y-b)\frac{d^2z}{dt^2} - (z-c)\frac{d^2y}{dt^2}\right\} = J.$$

If, as in Art. 209, we put $y = \bar{y} + \eta$, $z = \bar{z} + \zeta$, we get, by omitting the terms which vanish,

$$\mathfrak{M}\left\{(\bar{y}-b)\frac{d^2\bar{z}}{dt^2} - (\bar{z}-c)\frac{d^2\bar{y}}{dt^2} + k^2\frac{d\omega}{dt}\right\} = J. \qquad (4)$$

If we suppose the point b, c to coincide with the origin fixed in space, and to lie in the plane of the motion of the centre of inertia, this equation becomes, if we call r and χ the polar coordinates of the centre of inertia,

$$\mathfrak{M}\left\{\frac{d}{dt}\left(r^2\frac{d\chi}{dt}\right) + k^2\frac{d\omega}{dt}\right\} = J. \qquad (5)$$

241. **Moments of Momentum relative to any Point.**—Since the body is supposed to be moving parallel to a fixed plane, its motion at any instant is a pure rotation. If we take a line coinciding with the instantaneous axis of rotation as axis of x, then $\bar{x}$, $\bar{y}$, $\bar{z}$ being the coordinates of the centre of inertia, we have, by Art. 222,

$$\dot{\bar{x}} = 0, \quad \dot{\bar{y}} = -\bar{z}\omega, \quad \dot{\bar{z}} = \bar{y}\omega.$$

Substituting these values in (31), Art. 210, and introducing the values of H_1, H_2, H_3, given by (5), Art. 231, we obtain

$$\left.\begin{aligned} H'_1 &= \{I - \mathfrak{M}(b\bar{y} + c\bar{z})\}\omega, \\ H'_2 &= \{\mathfrak{M}a\bar{y} - \Sigma mxy\}\omega, \\ H'_3 &= \{\mathfrak{M}a\bar{z} - \Sigma mxz\}\omega \end{aligned}\right\}. \qquad (6)$$

EXAMPLES.

1. The motion of a body consists of a pure rotation; find the conditions that it should be brought to rest by a single impulse.

Take the axis of rotation as the axis of x, and a perpendicular p on it from the centre of inertia G as that of y, then the whole velocity of G is parallel to the axis of z, and is equal to $p\omega$, where ω is the angular velocity of the body. Hence the impulse which reduces the body to rest must be parallel to the axis of z, and is given by the equation

$$\underset{\cdot}{Z} = -\mathfrak{M}p\omega.$$

Let b, c be the coordinates of the point in which the impulse $\underset{\cdot}{Z}$ meets the plane of xy; the moments of momentum relative to bc are each zero after the body is reduced to rest; but, since the impulse passes through bc, these moments are the same as they were before the action of the impulse. Hence, originally,

$$H_1' = H_2' = H_3' = 0.$$

Substituting for H_1', &c., their values from (6), we have, if K be the radius of gyration round the axis of rotation,

$$K^2 - bp = 0, \quad \Sigma mxy = 0, \quad \Sigma mxz = 0.$$

Hence we conclude that the axis of rotation must be a principal axis at the point in which it is met by the perpendicular from G, that the impulse must be perpendicular to the plan containing G and the axis of rotation, and that its shortest distance b from the axis is given by the equation

$$bp = K^2. \qquad \text{(Compare Art. 235.)}$$

2. A uniform circular plate whose centre is fixed lies on a smooth horizontal plane. An insect starts from the centre of the plate, and returns to the same point after describing a circle whose diameter is the radius of the plate; find the angle through which the plate has turned.

Let ϕ be the angle through which the plate has turned at any time, a its radius, m its mass, m' that of the insect, r and θ its polar coordinates in space, r and ψ its polar coordinates relative to the plate; then

$$m\frac{a^2}{2}\frac{d\phi}{dt} + m'r^2\frac{d\theta}{dt} = 0, \quad \psi = \theta - \phi, \quad r = a\cos\psi.$$

Hence $$\phi = -\int \frac{2m'\cos^2\psi}{m + 2m'\cos^2\psi}\,d\psi,$$

and the angle required is

$$\pi\left\{1 - \left(\frac{m}{m+2m'}\right)^{\frac{1}{2}}\right\}.$$

242. **Equations of Motion for Impulses.**—In the case of impulses the changes of velocity which they produce are determined by the equations (Arts. 204, 209, 229),

$$\mathfrak{M}(v - v') = \Sigma \underset{\cdot}{Y}, \quad \mathfrak{M}(w - w') = \Sigma \underset{\cdot}{Z}, \quad \mathfrak{M}k^2(\omega - \omega') = \underset{\cdot}{L}, \quad (7)$$

where ω' is the angular velocity of the body, v' and w' the

components of the velocity of its centre of inertia, before the action of the impulses; and ω, v, and w the corresponding quantities after their action.

243. **Impact.**—When impact occurs between two *smooth* bodies, a mutual impulsive force is developed in the direction of the common normal. In the first period of collision this force reduces the relative normal velocity of the colliding points to zero. In the case of motion parallel to a fixed plane, there are for two bodies seven unknown quantities, viz. the changes in the two components of the velocity of the centre of inertia, and in the velocity of rotation for each body, and the magnitude of the mutual impulse. There are likewise seven equations to determine these quantities, viz. the six equations of motion, and the equation which expresses that the relative normal velocity of the colliding points is zero at the instant of greatest compression.

In the second period, a new mutual impulsive force is developed, whose impulse bears a constant ratio to that of the former, and can therefore be found. The changes of velocity which it produces can then be determined.

If the bodies which collide be perfectly elastic, the impulse developed during the period of restitution, or second period, is equal to that developed during the period of compression. What is here stated is merely a generalization of the theory given in Articles 78 and 202.

Examples.

1. A bar, which is rotating round an axis perpendicular to its length, and whose centre of inertia is moving in a plane at right angles to the axis of rotation, strikes perpendicularly against a fixed obstacle; determine the impulse of the blow, and the subsequent motion.

Let m be the mass of the bar, k its radius of gyration, V the velocity of its centre of inertia G in a direction perpendicular to its length, and Ω its angular velocity before impact; also let v' and ω', v and ω be the corresponding velocities at the end of the first and of the second period of impact, respectively; and let h be the distance from G of the point A at which the bar strikes the obstacle; then, if R' be the impulse of the blow during the first period of impact, and if we suppose the velocity of A due to the motion of translation to be in the same direction as that due to the rotation round an axis through G, we have, since the blow diminishes both the velocity of translation and the angular velocity of the bar,

$$R' = m(V - v'), \quad R'h = mk^2(\Omega - \omega');$$

but also, since at the instant of greatest compression A is reduced to rest, $v' + h\omega' = 0$. Hence we obtain

$$(k^2 + h^2)\,\omega' = k^2\Omega - hV;$$

therefore $\quad \Omega - \omega' = \dfrac{h(V + h\Omega)}{k^2 + h^2}$, whence $R' = \dfrac{mk^2(V + h\Omega)}{k^2 + h^2}$.

Now, as in Ex. 5, Art. 236, $R = (1 + e)R'$, and $\Omega - \omega = (1 + e)(\Omega - \omega')$, $V - v = (1 + e)(V - v')$. Hence we have $(k^2 + h^2)R = (1 + e)mk^2(V + h\Omega)$;

consequently $\quad v = \dfrac{(h^2 - ek^2)V - (1 + e)k^2h\Omega}{k^2 + h^2}, \quad \omega = \dfrac{(k^2 - eh^2)\Omega - (1 + e)hV}{k^2 + h^2}$.

2. Find the point at which the bar in Ex. 1 should strike the obstacle in order that the impulse of the blow should be a maximum.

We have here to determine h so that $\dfrac{V + h\Omega}{k^2 + h^2}$ shall be a maximum, and the required values of h are given by the quadratic equation $h^2 + 2\dfrac{V}{\Omega}h - k^2 = 0$, or $h^2 + 2rh - k^2 = 0$, if we put $r\Omega = V$. If C be the instantaneous centre of rotation of the bar, corresponding to V and Ω, we have $GC = -r$, and the points of the bar at which the impact produces the maximum impulse are determined by erecting a perpendicular GP equal to k, and with C as centre, and CP as radius describing a circle. The points A and B in which this circle meets the bar are the points required. (See Fig. p. 282.)

Let R_1 and R_2 be the values of the impulse R, corresponding to the points A and B, respectively, we readily find that

$$R_1 = \frac{1 + e}{2}\, m\Omega \,.\, BG, \text{ and } R_2 = -\frac{1 + e}{2}\, m\Omega \,.\, GA.$$

The negative sign of R_2 shows that in this case the impulse must act at the opposite side of the bar; hence, if we consider magnitude only, without regard to sign, each impulse may be regarded as a maximum.

3. A bar moving as in Ex. 1 strikes against a sphere of mass M, whose centre has a velocity U in a direction perpendicular to the bar; find the impulse of the blow, and the subsequent motion.

Let u' and u be the velocities of the centre of the sphere at the end of the first and of the second period of impact, then, the bar being supposed to overtake the sphere, we have

$$R' = m(V - v'), \; R'h = mk^2(\Omega - \omega'), \; R' = M(u' - U); \qquad (a)$$

and also, $$v' + h\omega' = u'. \qquad (b)$$

If we put $\Omega - \omega' = \varpi'$, from equations (a) we have

$$V - v' = \frac{k^2}{h}\varpi', \; u' - U = \frac{m}{M}\frac{k^2}{h}\varpi',$$

whence, substituting in (b) for v', u', and ω', we obtain

$$\varpi' = \frac{Mh(V + h\Omega - U)}{mk^2 + M(h^2 + k^2)}, \text{ and therefore } R = (1 + e)\,\frac{Mmk^2(V + h\Omega - U)}{mk^2 + M(h^2 + k^2)}.$$

Consequently the motion after collision is given by the equations

$$\omega = \Omega - (1 + e)\varpi', \quad v = V - (1 + e)\,\frac{k^2}{h}\,\varpi', \quad u = U + (1 + e)\,\frac{m}{M}\,\frac{k^2}{h}\,\varpi'.$$

4. Find in Ex. 3 at what point of its length the bar should strike the sphere in order that the impulse of the blow should be a maximum.

The values of h which make R a maximum are given by the quadratic equation

$$h^2 + 2rh - \left(\frac{M + m}{M}\right)k^2 = 0, \text{ where } r\Omega = V - U.$$

The points of the bar at which the impulse of the blow is a maximum may be determined by a construction similar to that of Ex. 2. In the present case, C is the point whose velocity perpendicular to the bar is equal to that of the sphere. The perpendicular to be erected at G is now $k\sqrt{\left(\frac{M + m}{M}\right)}$.

5. In Ex. 1 find the loss of kinetic energy due to the impact.

If $\mathcal{T}'$ be the kinetic energy lost during the first period of impact, we have, by Ex. 2, Art. 202, $2\mathcal{T}' = R'(V + h\Omega)$, but if $\mathcal{T}$ be the total loss of kinetic energy, $\mathcal{T} = (1 - e^2)\mathcal{T}'$ (see Ex. 4, Art. 202). Hence $2\mathcal{T} = (1 - e^2)\,\dfrac{mk^2(V + h\Omega)^2}{k^2 + h^2}$.

6. Find at what point the bar should strike the obstacle in order that the loss of kinetic energy should be a maximum. *Ans.* $h = \dfrac{k^2\Omega}{V}$.

7. In Ex. 3 find the total loss of kinetic energy of the system; and determine at what point the bar should strike the sphere in order that this loss should be a maximum.

Here, if $\mathcal{T}'$ be the kinetic energy lost by the system during the first period of impact, by Ex. 2, Art. 202,

$$2\mathcal{T}' = R'(V + h\Omega) - R'U,$$

hence

$$2\mathcal{T} = (1 - e^2)\,\frac{Mmk^2(V + h\Omega - U)^2}{mk^2 + M(h^2 + k^2)}.$$

This expression is a maximum when $h = \dfrac{m + M}{M}\,\dfrac{k^2\Omega}{V - U}$.

8. Find at what point the bar should strike the sphere in order that the gain of kinetic energy by the sphere should be a maximum.

The required points are those at which R is a maximum.

9. Find the loss of kinetic energy by the bar.

If $\mathcal{T}'$ be the loss of kinetic energy during the first period of impact, we have, Ex. 1, Art. 202,

$$2\mathcal{T}' = R'(V + h\Omega + v' + h\omega');$$

but $v' + h'\omega' = u' = U + \frac{R'}{M}$, and therefore, since $2\mathcal{T} = (1 - e^2)\mathcal{T}'$, we have

$$2\mathcal{T} = (1 - e^2)\frac{R'}{M}\{R' + M(V + h\Omega + U)\}$$

$$= (1 - e^2) Mmk^2 \left\{ mk^2 \left(\frac{V + h\Omega - U}{mk^2 + M\,(h^2 + k} \right)^2 + \frac{(V + h\Omega)^2 - U^2}{mk^2 + M(h^2 + k^2)} \right\}.$$

10. In Ex. 1 find the conditions that the whole motion of the bar should be destroyed by the collision.

> *Ans.* $k^2\Omega = hV$, and $e = 0$. This is also easily seen from first principles. See Ex. 1, Art. 241.

11. A body is moving parallel to a fixed plane, when a line AB in the body perpendicular to the plane becomes suddenly fixed; determine the subsequent motion.

Let m be the mass of the body, I its moment of inertia round AB, k its radius of gyration round a parallel axis through its centre of inertia G, Ω the angular velocity of the body, and V the velocity of G just before the line AB becomes fixed, p the shortest distance between the line of motion of G at this time and AB, and ω the angular velocity of the body round AB just after this line is fixed, then we have

$$I\omega = m(Vp + k^2\Omega).$$

12. A plane lamina is moving in its own plane when one of its points O becomes suddenly fixed; determine the subsequent motion.

Let us suppose that the lamina is constrained to rotate round a perpendicular axis through O, then, adopting the same notation as in Ex. 11, we have, by (10), Art. 234, since the axis of rotation is a principal axis at O,

$$I\omega = m(Vp + k^2\Omega),\ \dot{M}_0 = 0,\ \dot{N}_0 = 0.$$

Hence the actual motion of the lamina when O is fixed is a rotation round a perpendicular axis, and the angular velocity ω is given by the first of the equations above.

13. A bar moving in a vertical plane impinges upon a smooth horizontal plane; find the motion immediately after impact.

If the horizontal and vertical components of the velocity of the centre of inertia G of the bar be represented by U and V immediately before the impact, and by u and v immediately after, if Ω and ω be the corresponding angular velocities, a the distance from G to the point of impact of the bar, and α the

angle which it makes with the horizontal plane at the instant of impact, the values of v and ω are obtained by substituting in the equations of Ex. 1, $a \cos \alpha$ for h. Accordingly we have

$$u = U, \quad v = \frac{(a^2 \cos^2 \alpha - ek^2) V - (1 + e) k^2 a \Omega \cos \alpha}{k^2 + a^2 \cos^2 \alpha},$$

$$\omega = \frac{(k^2 - ea^2 \cos^2 \alpha) \Omega - (1 + e) a V \cos \alpha}{k^2 + a^2 \cos^2 \alpha}.$$

If the bar be homogeneous, $3k^2 = a^2$, and we get

$$v = \frac{(3 \cos^2 \alpha - e) V - (1 + e) a\Omega \cos \alpha}{1 + 3 \cos^2 \alpha},$$

$$\omega = \frac{(1 - 3e \cos^2 \alpha) a\Omega - (1 + e) V \cos \alpha}{(1 + 3 \cos^2 \alpha) a}.$$

14. In what direction must an impulse be applied to a sphere in order that its initial motion may be one of rotation round a given tangent?

The direction of the initial motion of the centre of inertia of the sphere is in this case given. Hence the direction of the impulse is a line parallel to this, lying in the plane, which passes through the centre, at right angles to the given tangent, and distant from the centre by $\frac{3}{5}$ radius.

15. A beam placed in a smooth horizontal plane is turning with a given velocity ω round a pivot which passes through a given point. The pivot breaks; determine the subsequent motion.

If b be the distance of the centre of inertia of the beam from the pivot, this point of the beam continues to move with a constant velocity $b\omega$ in the straight line which is at right angles to the beam at the moment when the pivot breaks, and the beam rotates with a constant angular velocity ω round a vertical axis through its centre of inertia.

16. A uniform bar, resting on a smooth horizontal table, revolves round a vertical axis through its middle point. The bar suddenly snaps at its middle point. Determine the subsequent motion of the parts.

17. In the same case, find the point of its length, at which either half of the bar would strike perpendicularly against a fixed obstacle with the greatest force of percussion.

18. Assuming that the Earth's orbit is circular, show that its motion, both of translation and of rotation, could be destroyed by a sudden impulse applied when the Earth is in a solstice.

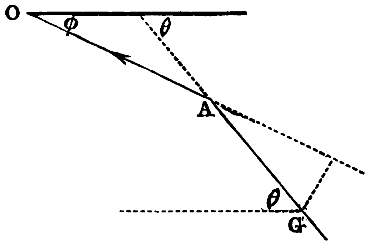
19. Assuming the Earth to be a homogeneous sphere, calculate in the preceding Example the distance from the Earth's centre of the line of action of the required impulse. *Ans.* 24 miles, approximately.

244. **Stress in Initial Motion.**—Stresses are determined, as we have seen, by using the dynamical equations for a free body, and introducing unknown reactions instead of the geometrical conditions. In many cases where the general equations of motion cannot be integrated, the initial stresses may be obtained by differentiating the geo-

metrical equations twice, and introducing into the equations thus obtained the initial values of the coordinates and of their differential coefficients with respect to the time, which are supposed to be given. The initial values of the accelerations are then in general determined, and thence the unknown reactions, by means of the dynamical equations.

Examples.

1. A lamina is suspended by strings attached to two of its points A and B, not in the same straight line with its centre of inertia, and fastened to two fixed points O and O'. The string joining O' to B is cut; determine the initial tension of the other.



The plane of the lamina in its position of equilibrium must pass through the points O and O', and the subsequent motion will take place in this plane, which we shall take as the plane of yz, the axis of y being horizontal, and the positive direction of z downwards, the origin being O. Let G be the centre of inertia of the lamina, ϕ and θ the angles which OA and AG make with the axis of y at any time, l and a the lengths of OA and AG, m the mass of the lamina, k its radius of gyration round a perpendicular axis through G, and y and z the coordinates of G; then, if T be the tension of the string OA at any time, we have

$$mk^2 \frac{d^2\theta}{dt^2} = -aT\sin(\theta - \phi), \qquad (a)$$

$$m\frac{d^2y}{dt^2} = -T\cos\phi, \quad m\frac{d^2z}{dt^2} = mg - T\sin\phi. \qquad (b)$$

Also, $$y = l\cos\phi + a\cos\theta, \quad z = l\sin\phi + a\sin\theta. \qquad (c)$$

Differentiating these latter equations twice, and in the second differentiation treating θ and ϕ as constants, since initially $\frac{d\theta}{dt}$ and $\frac{d\phi}{dt}$ are each zero, and finally substituting their initial values α and β for θ and ϕ, we obtain

$$-\frac{d^2y}{dt^2} = l\sin\beta\frac{d^2\phi}{dt^2} + a\sin\alpha\frac{d^2\theta}{dt^2}, \quad \frac{d^2z}{dt^2} = l\cos\beta\frac{d^2\phi}{dt^2} + a\cos\alpha\frac{d^2\theta}{dt^2}.$$

Hence, eliminating $\frac{d^2\phi}{dt^2}$, we get

$$\sin\beta\,\frac{d^2z}{dt^2} + \cos\beta\,\frac{d^2y}{dt^2} + a\sin(\alpha - \beta)\,\frac{d^2\theta}{dt^2} = 0.$$

Substituting from (*a*) and (*b*), and putting α and β for θ and ϕ in those equations, we get for T_0, the initial value of the tension,

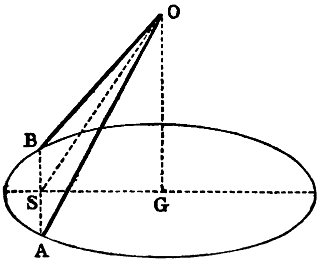
$$T_0 = mg\,\frac{k^2\sin\beta}{k^2 + a^2\sin^2(\alpha - \beta)}.$$

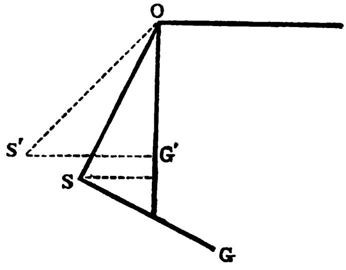
2. A body, whose centre of inertia is G, is suspended by strings attached to two of its points A and B, and fastened to two fixed points O and O'. The plane AGB is a principal plane at G, the string $O'B$ is cut; determine the initial tension of the other.

We may here suppose the body compelled to rotate round an axis through G, whose direction is fixed in space, and is perpendicular to the initial position of the plane AGB. Since this axis is a principal axis at G, we find, then, Art. 236, equation (14), that the components of the stress couple on this axis are zero, and therefore that the body rotates round it freely. Hence the whole motion of the body is parallel to the vertical plane which is the initial position of AGB, and the question becomes the same as in the last example.

3. A circular disk is hung, with its plane horizontal, from a fixed point vertically over its centre, by means of three equal strings attached to three fixed points in the circumference of the disk at equal distances from each other. One of the strings is cut; determine the initial tensions of the other two.

The two tensions along the threads OA and OB may be replaced by the single force F along OS, where $F = 2T\cos AOS$, S being the middle point of the chord joining the fixed points A and B.





In this case F takes the place of T, and the point S of A in Ex. 2. Then, β being the initial value of the angle which OS makes with the horizontal line which is the initial direction of SG, the length of the latter being a, we have, since SG is originally horizontal,

$$F = mg\,\frac{k^2\sin\beta}{k^2 + a^2\sin^2\beta}.$$

If l be the length of OS, the expression for F may be put into the form

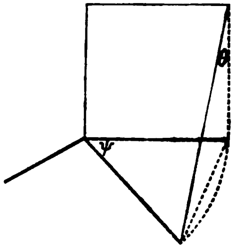
$$F = mg\frac{k^2 \sin\beta}{k^2 + l^2 \sin^2\beta\cos^2\beta}.$$

4. Determine in Ex. 9, Art. 202, the initial tensions of the strings, and their tensions when the bar is at its greatest height, the length of each string being $2a$.

If θ be the angle one of the strings makes with a vertical line at any time, z the vertical coordinate of the middle point of the bar, ψ the angle the bar makes with a horizontal line parallel to the fixed bar, and T the tension of one string; then

$$m\frac{a^2}{3}\frac{d^2\psi}{dt^2} = -2aT\sin\theta\cos\tfrac{1}{2}\psi,$$

$$m\frac{d^2z}{dt^2} = 2T\cos\theta - mg;$$



also, from the geometrical conditions,

$$\theta = \tfrac{1}{2}\psi, \text{ as } 2a\sin\theta = 2a\sin\tfrac{1}{2}\psi,$$

$$z = 2a(1-\cos\theta).$$

Substituting $\frac{1}{2}\psi$ for θ in the last equation, differentiating twice, and observing that initially $\psi = 0$, and $\dfrac{d\psi}{dt} = \omega$, we get for the initial tension of one string $T = \frac{1}{2}mg + \frac{1}{4}ma\omega^2$.

To get the tension when the bar is at its highest position, make $\dfrac{d\psi}{dt} = 0$, $\cos\frac{1}{2}\psi = \dfrac{2a-h}{2a}$, where h has the value in Ex. 9, Art. 202; then

$$T = mg\frac{4a^3}{(2a-h)\{4a^2+3h(4a-h)\}} = mg\frac{288g^3}{(12g-a\omega^2)\{48g^2+\omega^2(24ag-a^2\omega^2)\}}.$$

5. In Ex. 1, find the values of α and β, in order that T_0 shall be the greatest possible.

Ans. $\alpha = \beta = \dfrac{\pi}{2}$. The corresponding value of T_0 is mg, i.e. the weight of the lamina.

6. If β be given, find α, so that T_0 shall be a minimum.

Here $\sin(\alpha-\beta) = \text{max.}$, and therefore $\alpha - \beta = \dfrac{\pi}{2}$, or AG is perpendicular to OA.

7. If the initial position of AG be horizontal, find β, so that T_0 shall be a maximum.

Here we have to find β, so that $\dfrac{\sin\beta}{k^2} + \dfrac{1}{a^2\sin\beta}$ may be a minimum. Therefore, $k = a\sin\beta = p_0$, where p_0 is the initial value of the perpendicular from G on OA.

The result here obtained holds good for Ex. 3, if OS be substituted for OA, and F for T.

245. **Friction.**—Friction (*see* Art. 60) is a tangential force passing through the point of contact of two rough surfaces, which tends to prevent the one from slipping on the other. If there be slipping, the friction is in an opposite direction, and takes its greatest possible value, which is in a constant ratio to the normal pressure between the surfaces. If the motion be pure rolling, just enough friction is exerted to maintain pure rolling. The force of friction is then usually less than its maximum value, and is determined, as if it were an unknown reaction, by means of the equations of motion and the geometrical condition which expresses that the motion is pure rolling. If the value thus found for the force of friction does not exceed its maximum value, and pure rolling be consistent with the initial conditions, it will be the actual motion. When there is slipping, the friction, which is then a maximum, and therefore determined, tends to make the motion pure rolling. If pure rolling be attained, the friction at the instant pure rolling commences changes in general its value, and must be determined in the manner stated above.

It is to be observed, as already stated in Art. 60, that the maximum value of friction, when slipping actually takes place, is, in general, less than its maximum value when there is no slipping, and friction is acting against a force which tends to produce slipping.

When a surface is said to be *perfectly rough* it is understood that no slipping can take place between it and any other surface with which it is in contact. The amount of force which it is capable of exerting by means of friction is, in this case, unlimited.

Examples.

1. A homogeneous cylinder, having its axis horizontal, rolls without slipping down a rough inclined plane; determine the amount of friction brought into play (*see* Ex. 1, p. 139).

The equations of motion are

$$M\frac{d^2y}{dt^2}=Mg\sin i-F,\quad Mk^2\frac{d^2\theta}{dt^2}=aF;$$

the axis of y being a line in the inclined plane at right angles to its intersection

with the horizon. Also, $a d\theta = dy$; whence

$$F = Mg \frac{k^2 \sin i}{a^2 + k^2};$$

but since the cylinder is homogeneous we have $k^2 = \dfrac{a^2}{2}$,

and therefore $$F = \tfrac{1}{3} Mg \sin i.$$

2. If a sphere be substituted for a cylinder in the last example, determine the amount of friction brought into play. *Ans.* $F = \frac{2}{7} Mg \sin i$.

3. A lamina is placed on a rough horizontal table in such a manner that its centre of inertia lies beyond the edge of the table, and that the line in which the edge meets the lamina is a principal axis for the point O in which it is met by the perpendicular from the centre of inertia G; determine the motion of the lamina before it slips, and its inclination to the table when slipping begins.

Since the force tending to make the lamina slip is at first zero, the motion of the lamina begins by a rotation round the edge AB of the table, as a fixed axis.

Putting M for the mass of the lamina, and otherwise adopting the same notation as in Ex. 3, Art 236, we have, since $\alpha = 0$,

$$P_0 = Mg \frac{k^2 + 3p^2}{k^2 + p^2} \sin \theta, \quad Q_0 = Mg \frac{k^2}{k^2 + p^2} \cos \theta.$$

The lamina continues to rotate round AB till $P_0 = \mu Q_0$, where μ is the coefficient of friction. The value of θ when the lamina begins to slip is given therefore by the equation

$$\tan \theta = \frac{k^2}{k^2 + 3p^2} \mu.$$

4. In Ex. 3 a mass m is placed at a point D on the lamina, in the perpendicular from its centre of inertia on the edge of the table; investigate the motion, and find the inclination at which slipping begins.

Let $OD = h$, then, since the initial motion is a rotation round AB as a fixed axis, we have

$$\{M(k^2 + p^2) + mh^2\} \frac{d^2\theta}{dt^2} = (Mp + mh) g \cos \theta. \qquad (a)$$

Hence, by integration,

$$\omega^2 = \left(\frac{d\theta}{dt}\right)^2 = 2 \frac{Mp + mh}{M(k^2 + p^2) + mh^2} g \sin \theta. \qquad (b)$$

The forces acting on m, in addition to gravity, are the force of friction P along DO, and the resistance Q, perpendicular to DO, of the plane. Hence the accelerations of m, along DO, and perpendicular thereto, are

$$g \sin \theta - \frac{P}{m}, \text{ and } g \cos \theta - \frac{Q}{m};$$

but, since OD is invariable so long as m does not slide, the accelerations of m are also $-h\omega^2$, and $h\frac{d\omega}{dt}$, by (11) and (12), Art. 28; hence we have

$$P = m(g \sin\theta + h\omega^2), \quad Q = m\left(g \cos\theta - h\frac{d\omega}{dt}\right).$$

If we put
$$\lambda = \frac{(Mp + mh)\,h}{M(k^2 + p^2) + mh^2},$$

equations (a) and (b) give

$$h\frac{d\omega}{dt} = \lambda g \cos\theta, \text{ and } h\omega^2 = 2\lambda g \sin\theta;$$

hence we get
$$P = mg \sin\theta\,(1 + 2\lambda), \quad Q = mg \cos\theta\,(1 - \lambda).$$

We here observe that Q becomes negative if λ be greater than unity; accordingly in that case the mass m is *left behind* by the lamina from the very commencement of the motion, unless we place it *beneath* the lamina.

If $\lambda = 1$, or $ph = k^2 + p^2$, we have $Q = 0$: m is in this case placed at the centre of oscillation, and begins to slip at the very commencement of the motion. If $\lambda < 1$, m begins to slip when $\tan\theta_1 = \mu\frac{1 - \lambda}{1 + 2\lambda}$.

Next, let P_0 and Q_0 be the forces parallel and perpendicular to OG, exerted against the edge of the table, we have, (15), Art. 236, since the whole system at first is moving as a rigid body,

$$P_0 = (Mp + mh)\,\omega^2 + (M + m)\,g \sin\theta,$$

$$Q_0 = (M + m)\,g \cos\theta - (Mp + mh)\frac{d\omega}{dt}.$$

Hence, if $\nu = \frac{Mp + mh}{(M + m)\,h}$, we readily get

$$P_0 = (M + m)\,g \sin\theta\,(1 + 2\lambda\nu), \quad Q_0 = (M + m)\,g \cos\theta\,(1 - \lambda\nu).$$

If the coefficient of friction relative to the lamina be the same for the edge as it is for m, the lamina begins to slip when $P_0 = \mu Q_0$, or when

$$\tan\theta_0 = \frac{\mu(1 - \lambda\nu)}{1 + 2\lambda\nu}.$$

Hence, if $\nu < 1$, *i.e.* if $h > p$, we have $\theta_0 > \theta_1$, and therefore in this case the mass will slip before the lamina begins to slip.

On the other hand, if $h < p$, we have $\theta_1 > \theta_0$, and slipping begins at the edge AB.

5. In Ex. 3, if any number of masses m_1, m_2, &c., be placed on the lamina at points D_1, D_2, &c., on the line OG, investigate the motion.

Let $OD_1 = h_1$, $OD_2 = h_2$, &c., then,

$$\frac{d\omega}{dt} = \frac{Mp + m_1 h_1 + m_2 h_2 + \&c.}{M(k^2 + p^2) + m_1 h_1^2 + m_2 h_2^2 + \&c.} g \cos\theta,$$

$$\omega^2 = 2 \frac{Mp + m_1 h_1 + m_2 h_2 + \&c.}{M(k^2 + p^2) + m_1 h_1^2 + m_2 h_2^2 + \&c.} g \sin\theta.$$

If we put

$$\frac{Mp + m_1 h_1 + m_2 h_2 + \&c.}{M(k^2 + p^2) + m_1 h_1^2 + m_2 h_2^2 + \&c.} = \frac{\lambda_1}{h_1} = \frac{\lambda_2}{h_2} = \&c.,$$

$$\frac{Mp + m_1 h_1 + m_2 h_2 + \&c.}{M + m_1 + m_2 + \&c.} = \nu_1 h_1 = \nu_2 h_2 = \&c.,$$

we have

$$P_1 = (1 + 2\lambda_1) m_1 g \sin\theta, \quad Q_1 = (1 - \lambda_1) m_1 g \cos\theta;$$

$$P_2 = (1 + 2\lambda_2) m_2 g \sin\theta, \quad Q_2 = (1 - \lambda_2) m_2 g \cos\theta;$$

&c., &c.;

and also

$$\frac{P_0}{(M + m_1 + m_2 + \&c.) g \sin\theta} = 1 + 2\lambda_1\nu_1 = 1 + 2\lambda_2\nu_2 = \&c.,$$

$$\frac{Q_0}{(M + m_1 + m_2 + \&c.) g \cos\theta} = 1 - \lambda_1\nu_1 = 1 - \lambda_2\nu_2 = \&c.$$

The rest of the investigation is the same as in the last example.

If $h_1 > h_2$, then $\lambda_1 > \lambda_2$, and $\dfrac{1 - \lambda_1}{1 + 2\lambda_1} < \dfrac{1 - \lambda_2}{1 + 2\lambda_2}$,

and therefore $\theta_1 < \theta_2$, or m_1 slips before m_2: that is, the mass farthest from the edge begins to slip first.

6. If a hoop rolls down a rough inclined plane without sliding, show that $\tan i < 2\mu$; the initial position of the hoop being in a vertical plane at right angles to the intersection of the inclined plane with the horizon.

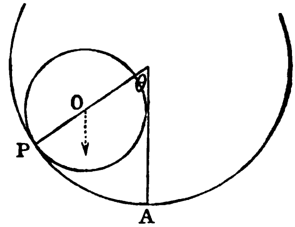
Take the initial position of the centre of the hoop for origin, and the intersection of the inclined plane with a vertical plane at right angles thereto as axis of y, its positive direction being downwards. Let the positive direction of rotation be from the upper side of the inclined plane towards y positive. Then, y being the coordinate of the centre of the hoop, m its mass, a its radius, and F the friction brought into play, the equations of motion are

$$ma^2 \frac{d\omega}{dt} = Fa, \quad m\frac{d^2 y}{dt^2} = mg \sin i - F;$$

but the motion being pure rolling, $a\omega = \frac{dy}{dt}$; hence, eliminating we obtain,

$$F = \frac{mg\sin i}{2}; \text{ but } F < \mu mg\cos i; \text{ therefore } \tan i > 2\mu.$$

7. A homogeneous circular disk, whose radius is a, rolls inside a rough vertical circle whose radius is b; the motion is pure rolling under the action of gravity; show that the rolling forward and backward of the disk is isochronous with the oscillations of a simple pendulum whose length is $\frac{3}{2}(b-a)$.



We have, I being the moment of inertia of the disk round an axis through P, ω the angular velocity, and θ the angle between CA and CP, $\frac{1}{\omega}\frac{d}{dt}(I\omega^2) = 2mga\sin\theta$ (θ being reckoned from the vertical line CA, where C is the centre of the vertical circle, and ω being the angular velocity of the disk rolling down). As the instantaneous centre of rotation lies, in this case, on the circumference of the disk, I remains constant throughout the motion; therefore $I\frac{d\omega}{dt} = mga\sin\theta$; but $I = \frac{3}{2}ma^2$ (*Integral Calculus*, Chap. X.), and $a\omega = -(b-a)\frac{d\theta}{dt}$, since either represents the velocity of O. Hence

$$\tfrac{3}{2}(b-a)\frac{d^2\theta}{dt^2} = -g\sin\theta; \therefore \text{ \&c.}$$

The student will observe that the friction at P does not enter this equation.

8. A uniform sphere, resting on a rough horizontal plane, is set in motion by an impulse applied in a vertical plane passing through its centre. Show that, when sliding ceases, the rolling motion will be direct, stationary, or retrograde, according as the direction of the impulse intersects the vertical diameter above, at, or below the point of contact with the plane.

Let v be the velocity, at any time, of the centre of the sphere parallel to the intersection of the horizontal plane with the vertical plane containing the impulse; the direction of the latter making an acute angle with the positive direction of v. Let ω be the angular velocity of the sphere, counted from the vertical towards the direction of v positive: then V and Ω, the initial values of v and ω, are determined by the equations

$$mV = Y, \quad mk^2\Omega = Yb,$$

where Y is the horizontal component of the impulse, and b the distance from the centre, at which its line of direction intersects the vertical diameter of the sphere. Eliminating Y, we obtain

$$\Omega = \frac{bV}{k^2}.$$

For the subsequent motion, if F be the force of friction, we have the equations

$$m\frac{dv}{dt} = F, \quad mk^2\frac{d\omega}{dt} = -Fa;$$

whence
$$a\frac{dv}{dt} + k^2\frac{d\omega}{dt} = 0.$$

Integrating, we obtain

$$av + k^2\omega = \text{constant} = aV + k^2\Omega = (a+b)V.$$

When sliding ceases $v = a\omega$. Substituting for v in the preceding equation, we have

$$\omega = \frac{(a+b)V}{a^2+k^2}.$$

Hence, since V is necessarily positive, ω, when sliding ceases, is positive, zero, or negative, according as

$$a+b>0, \quad a+b=0, \quad \text{or} \quad a+b<0.$$

The first condition holds good, if b is either positive, or negative and less than a in absolute magnitude; the second, if $b = -a$; the third, if b is negative and greater than a in absolute magnitude.

The results of this example may be extended to other solids of revolution.

9. A circular plate rolls down the inner circumference of a rough circle under the action of gravity. The plane of the plate coincides with that of the rough circle, which is vertical. Determine the amount of friction brought into play if the plate start from rest, the motion being pure rolling. (*See* Ex. 7.)

If ω be the angular velocity of the plate, the equations of motion are

$$\tfrac{1}{2}ma^2\frac{d\omega}{dt} = Fa, \quad m(b-a)\frac{d^2\theta}{dt^2} = -mg\sin\theta + F$$

together with the equation of condition

$$(b-a)\frac{d\theta}{dt} = -a\omega;$$

hence
$$F = \tfrac{1}{3}mg\sin\theta.$$

10. Show that the plate in the last example will ascend to the same height as that from which it started, and that the motion will go on for ever.

The *vis viva* $= 2mg(z - z_0)$: this will vanish when $z = z_0$; therefore, &c.

11. Determine the velocity of rotation of the plate at any time.

Ans. $\omega^2 = \frac{4}{3}\frac{(b-a)g}{a^2}(\cos\theta - \cos\theta_0)$.

246. **Tendency of a Rod to Break.**—When a body is under the influence of any forces, it experiences pressures or tensions, which tend to alter the relative positions of the molecules. This tendency is resisted by the mutual action of the molecules. Under such circumstances the body is said to be in a state of *stress*.

If we consider a small rectangular parallelepiped in the body, the stresses acting on one of its faces may be resolved into three forces at right angles to each other—one normal, and two parallel to the face under consideration.

To ascertain the tendency of a body to undergo a rupture in any part, we must consider the stresses to which it is subjected in that part. If the mutual cohesion of the molecules is unable to resist these stresses the body must give way. The question is, in general, one of great complication, and for its full discussion the reader is referred to treatises on *Elasticity* and *Strength of Materials*.

If the body under consideration be a *rod*, that is, if two of its dimensions are at each point very small, the question becomes much simplified.

The axis of the rod may be a straight line, or may form a curve of any kind. We shall suppose that this curve is not closed, that it lies in one plane P, and that the rod is in equilibrium under the action of forces in this plane. If we consider a section at right angles to the axis of the rod, at any point A of its length, the action of the molecules at one side of this section on those at the other must equilibrate all the forces acting on the rod at the latter side. These may be reduced to a force F, passing through A, and a couple G, round an axis a at right angles to the plane P. This force and couple, therefore, are equivalent to the stresses acting on the rod through the section containing A.

That the tendency of the rod to break results chiefly from the couple may be shown as follows:—

The stresses in the plane of the section cannot give any couple round the axis a, since a either meets them or is parallel to them. Hence the couple G must produce stresses, parallel to the axis of the rod at the point A, whose moment round A is equal to G. If N be the value per unit of area of the

greatest of these stresses, and a be the distance from A of the most remote point of the section, whose area may be denoted by S; the moment round A of the stresses parallel to the axis must be less than NSa. Hence, if we assume $G = Fp$, we have

$$NSa > Fp, \text{ and therefore } N > \frac{F}{S}\frac{p}{a}.$$

If we now seek for the stress per unit of area caused by the force F, we have $N' = \frac{F}{S}$; $\therefore\ N' < \frac{a}{p}N$.

Hence, if a is very small compared with p, N' is unimportant compared with N. Accordingly, in general, the tendency of the rod to break at any point A depends simply on N, *i.e.* on G, the moment round A of the forces acting on the rod at one side of A.

We have hitherto supposed the rod to be at rest. If it be in motion, we can, by D'Alembert's Principle, consider it as in equilibrium under the action of the applied forces and the forces of inertia, and the question of stress, or the tendency to break at any point, becomes the same as before, except that we must now add the forces of inertia to the other forces acting on the rod.

If the rod be acted on by impulses, the impulsive tendency to break at any point is obtained in a similar manner, and the preceding investigation holds good provided the impulses be substituted for the applied forces, and the resulting changes of momentum for the forces of inertia.

To find the couple which measures the tendency of a rod to break at any point P.

Let G be the required couple, L' the moment round P of the forces applied to the portion of the rod on one side of this point, $\mathfrak{M}'$ the mass of this portion of the rod, k' its radius of gyration round its own centre of inertia C', and Λ' the moment of the acceleration of C' round P, then by (4), Art. 240,

$$G = L' - \mathfrak{M}'\left(\Lambda' + k'^2\frac{d\omega}{dt}\right). \qquad (8)$$

In the case of impulses, if $\underset{\cdot}{G}$ be the impulsive couple corresponding to G, we have

$$\underset{\cdot}{G} = \underset{\cdot}{L}' - \mathfrak{M}' \{\underset{\cdot}{\Lambda}' + k'^2 (\omega - \omega')\}, \tag{9}$$

where $\underset{\cdot}{\Lambda}'$ is the moment round P of the change of velocity of C' due to the impulses, and ω and ω' are the angular velocities of the rod after and before the action of the impulses.

Another expression for G which is often useful may be found as follows:—Let $\bar{y}$, $\bar{z}$ be the coordinates, referred to a fixed origin, of the centre of inertia C of the whole rod; b, c those of P; y', z' those of C', and η, ζ those of any point of the rod referred to axes through C parallel to the fixed axes, then,

$$G = L' - \Sigma' m \{(y - b)\, \ddot{z} - (z - c)\, \ddot{y}\}.$$

But $$\ddot{y} = \ddot{\bar{y}} + \ddot{\eta}, \quad \ddot{z} = \ddot{\bar{z}} + \ddot{\zeta};$$

substituting, and remembering that

$$\Sigma' my = \mathfrak{M}' y', \quad \Sigma' mz = \mathfrak{M}' z',$$

we obtain

$$G = L' - \mathfrak{M}'\{(y'-b)\,\ddot{\bar{z}} - (z'-c)\,\ddot{\bar{y}}\} - \Sigma' m\{(y-b)\,\ddot{\zeta} - (z-c)\,\ddot{\eta}\}. \tag{10}$$

In the case of impulses applied to a rod at rest,

$$\underset{\cdot}{G} = \underset{\cdot}{L}' - \mathfrak{M}'\{y'-b)\,\dot{\bar{z}} - (z'-c)\,\dot{\bar{y}}\} - \Sigma' m\{(y-b)\,\dot{\zeta} - (z-c)\,\dot{\eta}\}. \tag{11}$$

If the rod be in motion when the impulses are applied, we must substitute in (11) for $\dot{\bar{y}}$, $\dot{\bar{z}}$, $\dot{\eta}$, and $\dot{\zeta}$ the changes in their values due to the action of the impulses.

Examples.

1. A uniform straight rod AB rotating round a perpendicular axis passing through one extremity A is struck perpendicularly at a point Q; find the tendency to break at any point P.

Let R be the impulse of the blow, a the length of the rod, m its mass, ω' and ω its angular velocities before and after the blow; also let C' be the middle point of PB, and let $AP = r$, $AQ = h$; then,

$$\underset{\cdot}{L}' = R \,.\, PQ, \quad \underset{\cdot}{\Lambda}' = PC' \,.\, AC' \,(\omega - \omega'), \quad 3k'^2 = PC'^2, \quad m'a = 2mPC',$$

hence we have

$$G = R \, . \, PQ - \frac{2m}{3a} PC'^2 (3AC' + PC')(\omega - \omega').$$

But
$$m \frac{a^2}{3} (\omega - \omega') = R \, . \, AQ = Rh,$$

$$PQ = h - r, \quad PC' = \frac{a - r}{2}, \quad AC' = \frac{a + r}{2}.$$

Substituting these values in the equation for G we obtain

$$G = \frac{R}{2a^3} \{2a^3 (h - r) - h (a - r)^2 (2a + r)\}$$

$$= \frac{Rr}{2a^3} \{a^2 (3h - 2a) - hr^2\}.$$

2. In Ex. 1 find the position of the point at which the tendency to break is a maximum.

If $3h > 2a$, the tendency to break is a maximum when

$$r = 2a \sqrt{\left\{1 - \frac{2a}{3h}\right\}}.$$

3. A uniform rod is turning in a vertical plane round a horizontal pivot A, at one of its extremities. Find the tendency to break at any point P.

Adopting the same notation as in Ex. 1, and denoting by θ the angle which the rod makes with the horizontal line, we have

$$L' = \frac{a - r}{2} m'g \cos \theta.$$

Moreover, since C' is moving in a circle round A as centre, its acceleration has two components—one at right angles to PC', which is

$$\frac{a + r}{2} \frac{d^2\theta}{dt^2},$$

and the other along PC'. The latter gives no moment round P; hence

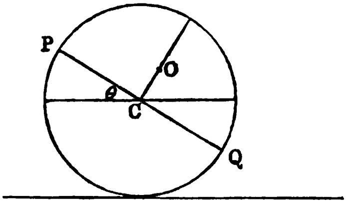
$$\Lambda' = \frac{a + r}{2} \, \frac{a - r}{2} \, \frac{d^2\theta}{dt^2},$$

and
$$G = \frac{a - r}{2} m'g \cos \theta - m' \left\{\frac{a^2 - r^2}{4} \frac{d^2}{dt} + k'^2 \frac{d^2\theta}{dt^2}\right\};$$

but
$$\tfrac{1}{3} ma^2 \frac{d^2\theta}{dt^2} = \tfrac{1}{2} mga \cos \theta, \text{ and } k'^2 = \frac{(a - r)^2}{12};$$

whence
$$G = - mg \frac{(a - r)^2}{4a^2} r \cos \theta.$$

4. A cracked hoop rolls on a perfectly rough horizontal plane. Determine the inclination to the horizon of the line joining the crack to the opposite point when the tendency to break at this point is the greatest possible.



In this case the centre of inertia of the hoop moves in a straight line with a constant velocity. Hence its acceleration is zero; also if a be the radius of the hoop,

$$L' = \frac{M}{2} g \left(a \cos\theta + \frac{2a}{\pi} \sin\theta \right), \text{ since } CO = \frac{2a}{\pi},$$

O being the centre of inertia of the semi-hoop comprised between the crack Q and the opposite point.

Now, since the angular velocity round a horizontal axis through C is constant, the system of forces $m\ddot{\eta}$, $m\ddot{\zeta}$, &c. in (10), are equivalent to a single force $M'\omega^2 . CO$ in the direction of CO. The moment of this force round P is $M\frac{\omega^2}{\pi} a^2$, which is independent of θ. The tendency to break at P is given by the equation

$$G = M \left\{ ag \left(\frac{\cos\theta}{2} + \frac{\sin\theta}{\pi} \right) - \frac{a^2\omega^2}{\pi} \right\}.$$

Hence the tendency to break is a maximum when $\tan\theta = \frac{2}{\pi}$, provided $g(2 + \sqrt{\pi^2 + 4}) > 4a\omega^2$. This condition appears by considering when G attains its greatest magnitude, irrespective of sign, if it should become negative.

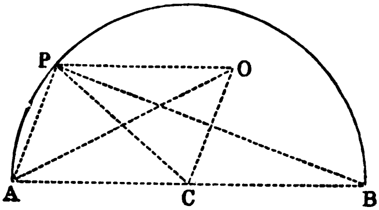
5. In Ex. 3 find at what point of the rod the tendency to break is a maximum. *Ans.* $r = \frac{1}{3}a$.

6. A semicircular wire, of radius a, lying on a smooth horizontal table, turns round one extremity A, with a constant angular velocity ω. Find the tendency to break at any point P.

Let O be the centre of inertia of the arc PB, and let $PCA = \phi$. Join AO, AP, and PO; then, since the angular velocity is constant, the acceleration of O is $\omega^2 . AO$. Consequently $\frac{\Lambda'}{\omega^2}$ is double the area of the triangle APO; but since AP and CO are parallel, the triangle APO is equal to the triangle ACP;

hence $\quad \Lambda' = a^2\omega^2 \sin\phi, \quad$ and $\quad G = m \frac{\pi - \phi}{\pi} a^2\omega^2 \sin\phi.$

Accordingly the tendency to break is a maximum at the point determined by the equation $\tan\phi = \pi - \phi$.



This example, as well as 3 and 4, are taken from Routh, *Rigid Dynamics*.

7. A free uniform straight rod is set in motion by a perpendicular impulse; find the tendency to break at any point P.

Let AB be the rod, C its middle point, M its mass, $2a$ its length, Q the point at which it receives the impulse R, C' the middle point of PB, M' its mass, v the velocity of C, and ω the angular velocity of the rod after the action of the impulse; let $CP = r$, $CQ = h$, then

$$L' = R\,.\,PQ, \quad (y'-b)\dot{z} - (z'-c)\dot{y} = PC'\,.\,v,$$

$$\Sigma' m\{(y-b)\dot{\zeta} - (z-c)\dot{\eta}\} = M'(PC'\,.\,CC'\omega + k'^2\omega);$$

but $$k'^2 = \frac{PC'^2}{3}, \quad PC' = \frac{a-r}{2}, \quad CC' = \frac{a+r}{2},$$

and also, $$Mv = R, \quad M\frac{a^2}{3}\omega = Rh;$$

hence from (11) we obtain by substitution

$$G = \frac{R}{4a^3}[4a^3(h-r) - (a-r)^2(a^2 + 2ha + hr)] = \frac{R}{4a^3}(r+a)^2(2ah - a^2 - hr).$$

8. In Ex. 7 find at what point of the rod the tendency to break is a maximum.

The value of r which makes $\frac{dG}{dr} = 0$ is $a\left(1 - \frac{2a}{3h}\right)$. If we substitute this value of r in G and in $\frac{d^2G}{dr^2}$, we find that G is positive and $\frac{d^2G}{dr^2}$ negative when $3h > a$; also G is negative and $\frac{d^2G}{dr^2}$ positive when $3h < a$. Hence in any case the tendency to break is a maximum when $r = a\left(1 - \frac{2a}{3h}\right)$.

247. **Impulsive Friction.**—When two *rough* surfaces collide, the investigation of what takes place is, in general, somewhat complicated. We must regard R and F, the impulses of the normal reaction and friction, as variable quantities, connected, at each instant of the impact, by linear equations with the coexisting values of the velocities of rotation of the bodies and of the velocities of translation of their centres of inertia. The laws which regulate the impulse of friction may then be stated as follows :—

(1) The direction of the elementary impulse dF due to friction is opposite to that of the slipping of the point of contact, if there be slipping; and if there be no slipping, is such as to prevent slipping.

(2) The magnitude of dF is, if possible, just sufficient to prevent slipping, and when slipping takes place $dF = \mu dR$, μ being the coefficient of dynamical friction.

The equations of motion for impulses (Art. 242) show that the relative normal and tangential velocities of the points of the bodies in contact are, at each instant, of the form $AR + BF + C$, where A, B, and C are constant during the impact.

The value of R is at first zero; when it becomes R_1 (at the end of the first period of the impact), the relative normal velocity is zero; and the maximum value of R, which it assumes at the end of the whole impact, is $(1 + e)R_1$.

These principles afford a sufficient number of equations to determine the motion; and, in the case of motion parallel to a fixed plane, the equations are always soluble.

If the bodies which collide are *perfectly rough*, the relative tangential velocity of the colliding points, or the velocity of slipping, is always zero; and when $R = R_1$, the relative normal velocity is likewise zero. Hence we have two equations to determine R_1 and the corresponding value of F. At the end of the impact $R = (1+e)R_1$; and the relative tangential velocity being still zero, the corresponding value of F can be determined.

If the bodies *slip on each other in the same direction during the whole of the impact*, dF is always equal to μdR; hence $F = \mu R$ throughout. R_1 is then determined from the equation expressing that the relative normal velocity is zero; and the

final values of R and F, which determine the motion after the impact, are $(1+e)R_1$ and $\mu(1+e)R_1$.

For a discussion of the problem in more complicated cases the reader is referred to Routh, *Rigid Dynamics.*

If a *sphere* impinges against a *fixed* surface, or if *two spheres* collide with each other, the relative tangential velocity v depends upon the velocities of rotation of the spheres, and the velocities of their centres parallel to the common tangent. It is therefore independent of the normal reaction, and the relative normal velocity in like manner is independent of the friction. In this case, if v become zero it must remain zero, as friction cannot initiate a relative tangential velocity in its own line of direction. Hence v must be either zero at the end of the impact, or in the same direction as at the beginning. Moreover, the value of R_1 is independent of F. The problem is, therefore, reducible to one of the two cases treated above.

If we assume at first that there is no slipping, and obtain the final value of F on this hypothesis, the solution is correct, provided the value of F so obtained does not exceed $\mu(1+e)R_1$. If this value of F does exceed $\mu(1+e)R_1$, then slipping takes place in the same direction throughout the impact, and the final value of F which determines the subsequent motion is $\mu(1+e)R_1$.

Examples.

1. A box, placed on a rough horizontal table, carries two vertical rods which support a horizontal rod from which a mass m is suspended. A fine string, fastened to the box, and passing over a pulley at the edge of the table, is attached to a mass M' which, when set in motion, causes the box and suspended mass m to move with a uniform velocity. The string which supports m is now cut, and m falls into the box. If its velocity after m has struck it be equal to its original velocity, and if the friction on the axle of the pulley be neglected, show that the coefficients of impulsive and continuous friction are equal.

Let M be the mass of the box and frame-work, v' its original velocity, μ the coefficient of dynamical friction, R the impulse of the normal reaction, and F the impulse of the friction, developed between the table and box when the latter is struck by m. Since the box originally moves with a constant velocity, we have $M'g = \mu(M+m)g$. After the string supporting m is cut, the box is acted on by an acceleration f, during the time t, in which m is falling. If I be the moment of inertia of the pulley, and a its radius, f is given by the equation

$$\left(M' + M + \frac{I}{a^2}\right) f = (M' - \mu M)g = \mu m g. \qquad (a)$$

The velocity of the box when struck by m is $v' + ft$. Hence its velocity v after the impact is given by the equation

$$\left(M + m + M' + \frac{I}{a^2}\right) v = \left(M + M' + \frac{I}{a^2}\right)(v' + ft) + mv' - F. \qquad (b)$$

If $v = v'$, from (b) we get $\left(M + M' + \frac{I}{a^2}\right) ft = F$; this, by ($a$), is reduced to $F = \mu mgt$; but $R = mgt$, and therefore $F = \mu R$.

This example is a description of the experiment by which Morin showed that impulsive and continuous friction obey the same law, and have the same coefficient.

If the friction on the axle of the pulley were taken into account, the terms arising from thence in the above equations would each contain as a factor the quantity $\frac{\alpha}{a}$, where α is the radius of the axle. But as α is very small compared with a, these terms may be neglected.

2. A sphere, rotating with an angular velocity Ω round a horizontal axis at right angles to the plane of the trajectory of its centre, impinges on a perfectly rough horizontal plane: find the motion immediately after impact.

Suppose the sphere is moving from left to right before impact with a velocity V, whose direction makes an angle i with the plane of the horizon. Let ω be the angular velocity in the direction of the motion of the hands of a watch, and v the horizontal velocity of the centre at the instant after impact. F being the impulse arising from friction, the equations of motion are

$$Mv = MV \cos i + F,$$

$$\tfrac{2}{5} Ma^2\omega = \tfrac{2}{5} Ma^2\Omega - aF.$$

The geometrical condition for no slipping is

$$v - a\omega = 0;$$

whence
$$\frac{F}{M} = -\tfrac{2}{7}(V \cos i - a\Omega),$$

$$v = a\omega = \tfrac{5}{7} V \cos i + \tfrac{2}{7} a\Omega.$$

If $V \cos i = a\Omega$, no impulsive friction is called into play. If $V \cos i > a\Omega$, the horizontal velocity of the centre of the sphere is diminished, and the sphere rebounds at a greater angle than if there were no friction. If $V \cos i < a\Omega$ the horizontal velocity of the sphere is increased, and the sphere rebounds at a smaller angle than if there were no friction. In this case friction accelerates the horizontal velocity of the centre of the sphere.

If Ω is opposite in direction to the motion of the hands of a watch,

$$v = \tfrac{5}{7} V \cos i - \tfrac{2}{7} a\Omega.$$

The velocity of the centre of the sphere along the horizontal line is diminished, and the sphere will rebound at a greater angle than if there were no friction. If $5V \cos i = 2a\Omega$ the sphere will rebound vertically. If $2a\Omega > 5V \cos i$ the sphere will hop back. This explains the effect of slow under-cut in tennis. The numerical factors for a tennis ball may of course be different from those given above.

The magnitude of the total normal reaction between the sphere and the plane is $M(1+e)V\sin i$. Hence, in any case in which $\mu > \dfrac{2(V\cos i - a\Omega)}{7(1+e)V\sin i}$, the preceding investigation holds good, even though the plane be not perfectly rough. If Ω be counter-clockwise its sign must be changed in the above expression for the limiting value of μ.

3. If the plane in the last example be imperfectly rough, so that the impulsive friction is not sufficient to destroy the whole tangential velocity of the point of contact of the sphere with the plane, determine the motion.

The equations are, if $V\cos i > a\Omega$,

$$Mv = MV\cos i - \mu(1+e)MV\sin i,$$

$$\tfrac{2}{5}Ma^2\omega = \tfrac{2}{5}Ma^2\Omega + \mu(1+e)MVa\sin i.$$

The sign of μ must be changed in these equations if $V\cos i < a\Omega$, and the sign of Ω if its direction be counter-clockwise.

248. **Rolling and Twisting Friction.**—In questions relating to friction, if great accuracy be required in the determination of the motion, it is necessary to take into account not only the tangential force of friction, but also what is called *the couple of rolling friction*, which is a couple having for its axis the tangent to the rough surface round which the body is rotating. Its maximum value is the normal pressure multiplied by a linear constant, and is generally small in amount, so that in solving questions connected with friction this couple is usually neglected. The direction in which the couple of rolling friction tends to turn the body is opposite to that in which it is actually rotating. If the body be not actually rotating, but be acted on by forces tending to make it rotate, the couple of rolling friction tends to prevent rotation round a common tangent to the two rough surfaces.

If the surfaces have a relative angular velocity about the common normal, then, besides the tangential force of friction, and the couple of rolling friction, there is also a couple, having the normal as its axis, called *the couple of twisting friction*. This couple likewise is usually small in amount.

Examples.

1. Taking into account the couple of rolling friction, and supposing the motion to be still pure rolling, determine in Ex. 9, Art. 245, the amount of friction brought into play, and the angular velocity in any position.

Let R be the normal reaction between the plate and circle at any time, and fR the couple of rolling friction; then the equations of rotation of the plate are

$$\tfrac{1}{2}ma^2\frac{d\omega}{dt} = Fa - fR, \quad m(b-a)\frac{d^2\theta}{dt^2} = F - mg\sin\theta.$$

Also $R = m\{g\cos\theta + (b-a)\dot{\theta}^2\}$, and $(b-a)\dot{\theta} = -a\omega$.

Hence, putting $\dfrac{f}{a} = \nu$, $\quad F = \tfrac{1}{3}m\left\{g\sin\theta + 2\nu g\cos\theta + 2\nu(b-a)\dot{\theta}^2\right\}$;

consequently we obtain

$$\tfrac{3}{2}(b-a)\frac{d^2\theta}{dt^2} = \nu\left\{g\cos\theta + (b-a)\dot{\theta}^2\right\} - g\sin\theta.$$

If we change the independent variable by means of the symbolic equation

$$\frac{d}{dt} = \dot{\theta}\frac{d}{d\theta}, \text{ and put } \frac{g}{b-a} = n, \text{ we get}$$

$$\frac{d}{d\theta}\left(\dot{\theta}^2\right) = \tfrac{4}{3}\left\{n(\nu\cos\theta - \sin\theta) + \nu\dot{\theta}^2\right\}.$$

The solution of this differential equation is of the form

$$\dot{\theta}^2 = Ce^{\frac{4}{3}\nu\theta} + D\cos\theta + E\sin\theta,$$

where C is an arbitrary constant. Determining the constants D and E, we obtain

$$\dot{\theta}^2 = Ce^{\frac{4}{3}\nu\theta} + \frac{4n}{9+16\nu^2}\left\{(3-4\nu^2)\cos\theta + 7\nu\sin\theta\right\}.$$

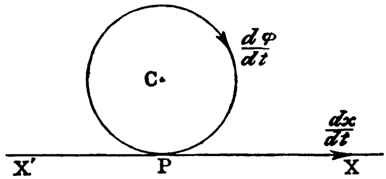
If θ_0 be the initial value of θ, we have, since $\dot{\theta}_0 = 0$,

$$C = -\frac{4n}{9+16\nu^2}\left\{(3-4\nu^2)\cos\theta_0 + 7\nu\sin\theta_0\right\}e^{-\frac{4}{3}\nu\theta_0}.$$

When $\dot{\theta}$ is determined, ω can be found by the equation $a\omega = -(b-a)\dot{\theta}$.

2. A circular plate is projected along a rough horizontal plane, with an initial velocity V of translation, and an angular velocity Ω, round an axis through its centre, at right angles to its plane. Determine the motion, neglecting the couple of rolling friction.

Let ω denote the angular velocity $\dfrac{d\phi}{dt}$, and v the velocity of the centre, at any time, and let x, the horizontal coordinate of the centre, be measured in the direction of V, as in the figure; then the whole velocity of P is $v - a\omega$, where a is the radius of the plate.



Different phenomena present themselves according to the values of V and Ω

(1) Ω positive, and $V > a\Omega$.

Since $V - a\Omega$ is positive, P begins to slip along PX; therefore $F = \mu Mg$, and the equations of motion are

$$M\frac{d^2x}{dt^2} = -\mu Mg, \quad \tfrac{1}{2}Ma^2\frac{d\omega}{dt} = \mu Mga.$$

Pure rolling commences when $v - a\omega = 0$, *i.e.* at a time t_0 equal to $\dfrac{V - a\Omega}{3\mu g}$;

then $$a\omega = v = \tfrac{2}{3}V + \tfrac{1}{3}a\Omega.$$

The equations for the subsequent motion are

$$v = a\omega, \quad M\frac{dv}{dt} = -F, \quad \tfrac{1}{2}Ma^2\frac{d\omega}{dt} = Fa,$$

where F is the amount of friction brought into play.

Hence $F = 0$, and the disk will roll on with a constant velocity of rotation round the instantaneous axis.

(2) Ω positive as before, $V < a\Omega$.

Since $V - a\omega$ is negative, P commences by slipping back towards X'.

The equations given above must in this case be modified by changing the sign of μ. The initial velocity of translation of the centre is, in this case, increased.

(3) Initial angular velocity negative and equal to $-\Omega$.

Here we must change the sign of Ω in the equations of case (1).

If $a\Omega > 2V$, both v and ω will be negative, that is, the motion of translation of the centre will be in the direction opposite to that originally imparted, and the rotation will be in the same direction as the initial rotation.

3. Discuss the same problem, taking into account the couple of rolling friction.

Here we have

$$M\frac{d^2x}{dt^2} = -\mu Mg, \quad \tfrac{1}{2}Ma^2\frac{d\omega}{dt} = \mu Mga - fMg.$$

Putting $\dfrac{f}{a} = \nu$, we find then that pure rolling commences when

$$t = \frac{V - a\Omega}{(3\mu - 2\nu)g} = t_0.$$

At this instant $$v = \frac{2V(\mu - \nu) + \mu a\Omega}{3\mu - 2\nu} = v_0.$$

After this the equations of motion become

$$M\frac{d^2x}{dt^2} = -F, \quad \tfrac{1}{2}Ma^2\frac{d\omega}{dt} = Fa - fMg,$$

along with $v = a\omega$; whence $F = \tfrac{2}{3}\nu Mg$.

This expression shows that the friction brought into play varies inversely as the radius of the plate, provided its mass be constant.

The plate will come to rest at a time

$$t' = \frac{3v_0}{2\nu g},$$

where t' is counted from the instant when pure rolling begins.

In order that the motion should become pure rolling it is necessary that $\mu > \frac{2}{7}\nu$.

The student will have no difficulty in investigating cases (2) and (3) of Ex. 2, when the couple of rolling friction is taken into account.

4. A sphere is projected down a rough inclined plane, along a line of greatest slope of the plane. The sphere has an initial velocity of rotation round a horizontal axis parallel to the inclined plane; determine the motion—(1) neglecting the couple of rolling friction; (2) taking that couple into account.

Let the line of projection be the axis of x, and let x positive be measured to the right, and ω, the angular velocity, be in the direction of the motion of the hands of a watch. Let V be the initial velocity of translation of the centre of the sphere, and Ω the initial angular velocity.

(1) The equations of motion are,

$$M\frac{d^2x}{dt^2} = Mg\sin i - F, \quad \tfrac{2}{5}Ma^2\frac{d\omega}{dt} = Fa,$$

and the condition for pure rolling is

$$v - a\omega = 0.$$

If t_0 be the time at which pure rolling begins, then

$$t_0 = \frac{2(V - a\Omega)}{(7\mu\cos i - 2\sin i)g}, \text{ or} = \frac{2(a\Omega - V)}{(7\mu\cos i + 2\sin i)g},$$

according as $V > a\Omega$, or $a\Omega > V$, where μ is the coefficient of *dynamical* friction. If $V - a\Omega > 0$, we must have $7\mu\cos i > 2\sin i$ in order that pure rolling should be attainable. If $V - a\Omega = 0$, pure rolling will continue, provided $7\mu'\cos i > 2\sin i$ (where μ' is the coefficient of *statical* friction). If $V - a\Omega < 0$, pure rolling will be reached necessarily, and will then continue, provided $7\mu'\cos i > 2\sin i$.

If v_0 and ω_0 be the values of v and ω when pure rolling is attained,

$$a\omega_0 = v_0 = \frac{5\mu V\cos i - 2(\sin i - \mu\cos i)a\Omega}{7\mu\cos i - 2\sin i},$$

or

$$a\omega_0 = v_0 = \frac{5\mu V\cos i + 2(\sin i + \mu\cos i)a\Omega}{7\mu\cos i + 2\sin i},$$

according as $V - a\Omega$ is positive or negative. It may be observed that the equations for the latter case can be obtained from those for the former by changing the sign of μ. After pure rolling begins, if it continues, $F = \frac{2}{7}Mg\sin$

$$v = a\omega = \tfrac{5}{7}(t - t_0)g\sin i + a\omega_0.$$

(2) The equations of motion are

$$M\frac{d^2x}{dt^2} = Mg\sin i - F, \quad \tfrac{2}{5}Ma^2\frac{d\omega}{dt} = Fa - fMg\cos i.$$

Hence, putting $\frac{f}{a} = \nu$, we have, when $V > a\Omega$, Ω being positive,

$$t_0 = \frac{2(V - a\Omega)}{\{7\mu - 5\nu)\cos i - 2\sin i\} g};$$

and in order that pure rolling may be possible, $7\mu - 5\nu > 2\tan i$.

Again, $$v_0 = a\omega_0 = \frac{5(\mu - \nu) V\cos i - 2(\sin i - \mu\cos i) a\Omega}{(7\mu - 5\nu)\cos i - 2\sin i},$$

and at any time after pure rolling is established,

$$a\omega = a\omega_0 + \tfrac{5}{7} g(\sin i - \nu\cos i)(t - t_0).$$

When $V < a\Omega$, the equations corresponding to this case are obtained from those above by changing the sign of μ.

If the initial angular velocity be negative, and equal to $-\Omega$, the equations of motion are

$$M\frac{d^2x}{dt^2} = Mg\sin i - \mu Mg\cos i,$$

$$\tfrac{2}{5} Ma^2 \frac{d\omega}{dt} = a\mu Mg\cos i + f Mg\cos i,$$

until $\omega = 0$. This takes place at a time t_1 given by the equation

$$t_1 = \frac{2a\Omega}{5g\cos i(\mu + \nu)},$$

Then

$$v_1 = \frac{2a\Omega(\sin i - \mu\cos i) + 5V\cos i(\mu + \nu)}{5\cos i(\mu + \nu)}.$$

After this ω is positive, and

$$t_0 = t_1 + \frac{2v_1}{\{(7\mu - 5\nu)\cos i - 2\sin i\} g} = 2\frac{(\mu - \nu)a\Omega + (\mu + \nu)V}{(\mu + \nu)\{(7\mu - 5\nu)\cos i - 2\sin i\} g},$$

and $$v_0 = a\omega_0 = \frac{\mu - \nu}{\mu + \nu} \cdot \frac{5(\mu + \nu) V\cos i + 2(\sin i - \mu\cos i) a\Omega}{(7\mu - 5\nu)\cos i - 2\sin i}.$$

5. A number of spheres are projected in different directions with different initial velocities along a rough horizontal plane; find the path of their common centre of inertia.

Ans. A series of parabolas, and finally a straight line (see (1), Ex. 2).

6. A hollow cylinder filled with water is projected without initial rotation in a direction perpendicular to its axis, along a rough horizontal plane; determine the time at which pure rolling begins, the amount of friction subsequently brought into play, and the time at which the cylinder comes to rest.

Let M be the mass of the cylinder and contained water, I the moment of inertia of the cylinder round its central axis, a the radius of its external surface, μ and f the coefficients of sliding and rolling friction, V the initial velocity of the common centre of inertia G of the cylinder and contained water, t_1 the time at which pure rolling begins, F the friction subsequently brought into play, v_1 the velocity of the point G at the time t_1, t_2, the time at which the cylinder comes to rest. Then, putting $\frac{f}{a} = \nu$, we find

$$t_1 = \frac{1}{\mu + \lambda(\mu - \nu)}\frac{V}{g}, \quad F = \frac{\lambda}{1+\lambda}\nu Mg, \quad v_1 = \frac{\lambda(\mu - \nu)}{\mu + \lambda(\mu - \nu)}V, \quad t_2 = \frac{V}{\nu g},$$

where $\lambda I = Ma^2$. As λ increases, F increases, and so in general does v_1, whilst t_1 diminishes, and t_2 in every case remains constant, being the same as in the case of a solid cylinder (see Ex. 3).

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CHAPTER XI.

MOTION OF A RIGID BODY IN GENERAL.

Section I.—*Kinematics.*

249. **Motion of a Body having one Point fixed.**—If a rigid body have a fixed point, a spherical surface S fixed in the body, with this point as centre, must move about on the surface of an equal concentric sphere fixed in space. The position in space of S, or of any definite great circle on it, determines that of the body. Hence the motion of a body having a fixed point is reducible to the motion of a spherical figure on a sphere fixed in space. The position of such a figure is determined by the positions of any two definite points A and B in it. If the points A and B move into new positions A' and B', arcs of great circles bisecting AA' and BB' at right angles will meet in a point O, and the angle $AOA' = BOB'$; but the great circle OA can be moved into the position OA' by turning it through the angle AOA' round the axis CO (C being the centre of the sphere); and since $AOA' = BOB'$, the same rotation brings OB into the position OB'. Hence a rotation round OC brings the spherical figure, of which A and B are definite points, from the first position into the second. The point O is called the pole of rotation (*Differential Calculus*, Art. 300).

Consequently, *a rigid body having a point fixed can be moved from any one position into any other by a rotation round an axis through the point.*

250. **Composition of Rotations round Axes meeting in a Point.**—If a body receive rotational displacements round two axes fixed in space, passing through the same point, the resultant displacement may be effected by a rotation round a single axis.

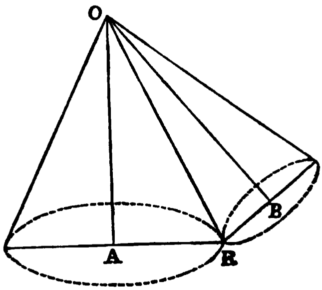
If the displacements be infinitely small, it appears, as in Article 220, that the order in which they are effected is in-

different, and also that it is indifferent whether the axes be fixed in space or be axes fixed in the body, whose positions at the commencement of the infinitely small motion coincide with those of the axes fixed in space. If the two displacements be regarded as simultaneous, the resultant rotation is the actual motion of the body. Hence we see that—

A velocity of rotation round a single axis is equivalent to velocities of rotation round two axes meeting the axis of the resultant rotation in the same point.

Beiny given the velocities of rotation of a rigid body round two axes meeting in a point, to determine the velocity of the resultant rotation and the position of its axis.

Let OA and OB be the axes of the component rotations, and R a point on the axis of the resultant rotation. As R is at rest during the motion, its displacement from the rotation round OA must be equal and opposite to that from the rotation round OB. Hence the circles passing through R, and having their planes at right angles to OA and OB, and their centres on those lines, touch at R.



Hence OA, OB, and OR lie in the same plane. This appears readily from the fact, that if two small circles of a sphere touch, the arc of a great circle joining their poles passes through the point of contact. Again, AR multiplied by the angular velocity round OA is equal and opposite to BR multiplied by the angular velocity round OB. If these angular velocities be denoted by a and β, we have

$$\frac{a}{\sin BOR} = \frac{\beta}{\sin AOR}.$$

To find ω, the angular velocity of the resultant rotation, consider the motion of A. It is unaffected by the rotation round OA, and may be regarded indifferently, as rotating round OB with angular velocity β, or as rotating round OR with angular velocity ω.

If perpendiculars AP and AQ be let fall on OB and OR, we have then $AP.\beta = AQ.\omega$. Hence

$$\frac{\omega}{\sin AOB} = \frac{\beta}{\sin AOR} = \frac{a}{\sin BOR}.$$

Hence, finally—*The axis of the resultant rotation lies in the same plane as the axes of the component rotations, and makes with each an angle whose sine is proportional to the velocity of rotation round the other; and the velocity of the resultant rotation is proportional to the sine of the angle between the axes of the component rotations.*

Accordingly, velocities of rotation are compounded in precisely the same manner as velocities of translation, or as forces meeting in a point.

By reversing the reasoning above, it can be shown that a point R, taken as above, remains at rest under the influence of two velocities of rotation round OA and OB; whence we have an independent proof, that infinitely small rotations round two intersecting axes are equivalent to a single one round an axis lying in the plane of the two former, and passing through their point of intersection.

We have already seen, Article 221, that velocities of rotation round parallel axes are compounded in the same way as parallel forces. Hence, in general—*Velocities of rotation are compounded like forces, whose directions coincide with the axes of rotation, and whose magnitudes are proportional to the velocities of rotation.*

The attention of the reader has been directed in Article 221 to the algebraical signs of velocities of rotation. In addition to what was there stated, it may be observed, that the axis of a rotation may be made to represent the rotation both in magnitude and direction. In this case the axis is drawn so that the rotation round it is always positive. For example, instead of speaking of a negative rotation round the axis of X, we may designate it simply as a rotation round the axis of X negative. When the axis of a rotation determines the direction of the rotation, the latter is always understood to be in the positive direction round this axis, that is, according to the convention, counter-clockwise.

When rotations are compounded by means of their axes, like forces, the direction of the axis determines in this way the direction of the rotation.

For example, rotations ω_1, ω_2, ω_3 round three rectangular axes produce a resultant rotation ω which is always positive; but the direction of its axis is determined by $\frac{\omega_1}{\omega}$, $\frac{\omega_2}{\omega}$, $\frac{\omega_3}{\omega}$, the cosines of the angles made with the coordinate axes; and these again depend on the signs of ω_1, ω_2, and ω_3, as well as on their magnitudes.

251. **Geometrical representation of the Motion of a Body having a Fixed Point.**—When a body has a fixed point, its motion may be represented in a manner analogous to that mentioned in Article 225. In the present case the curves which correspond to the space centrode and the body centrode are spherical curves lying on the surface of the same sphere.

The motion of the body is represented by the rolling of a cone fixed in the body on a cone fixed in space (see *Differential Calculus*, Article 301).

252. **Motion of a Body which is entirely Free.**—*A rigid body can be moved from any one position into any other by a motion of translation, combined with a motion of rotation round an axis through any arbitrary point A of the body.*

Let A_1, A_2 be the two positions in space occupied by A in the different positions of the body. Give to every point of the body a motion equal and parallel to $A_1 A_2$: this brings A into the required position, and a rotation round an axis through A will then (Article 249) complete the body's change of place.

If two positions of a body in motion are infinitely near each other, any infinitely small displacements, by which it can be moved from the first of these positions to the second, may be regarded as the actual motion of the body.

The actual motion of a rigid body during an infinitely short time is, therefore, a motion of translation together with a motion of rotation round an axis through any arbitrary point of the body.

The initial and final positions of a body being given, the magnitude of the rotation, which is required to make it pass from one to

the other, and the direction of its axis are determined; but the motion of translation varies according to the point through which the axis of rotation is supposed to pass.

First, let the axis of rotation be supposed to pass through a point A, whose initial and final positions are A_1, A_2. The motion of translation A_1A_2 is composed of two parts—one A_1A' in the direction of the axis of rotation through A, and the other $A'A_2$ at right angles to it. By means of the first a definite plane section of the body, passing through A and at right angles to the axis of rotation, is moved into the plane in space in which it lies in its final position, and the subsequent motion of the body is therefore parallel to this plane. If, now, the axis of rotation be regarded as passing through another point B of the body, whose initial and final positions are B_1, B_2, we can suppose the translation B_1B_2 made up of two parts—one, B_1B', equal and parallel to A_1A'; the other, $B'B_2$, which depends on the position of the point. B_1B' brings the body into the same position as A_1A'. Hence, a translation $B'B_2$ and a rotation round an axis through B are equivalent to an equal rotation round a parallel axis through A and a translation $A'A_2$ (Art. 219). The translation B_1B_2 is the resultant of B_1B' and $B'B_2$; A_1A_2 is the resultant of A_1A' and $A'A_2$; B_1B' is equal and parallel to A_1A'; but $B'B_2$ is not in general either equal or parallel to $A'A_2$.

253. **Analytical Treatment of the Motion of a Body having a Fixed Point.**—Suppose three rectangular axes fixed in the body passing through a point O; and three others fixed in space, which at the beginning of the motion coincide with the former. Let the coordinates of any point of the body referred to the former be ξ, η, ζ, and referred to the latter, x, y, z. Let a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3 be the cosines of the angles which ξ, η, ζ make with x, y, z, respectively; and let the angles themselves be α_1, α_2, α_3; β_1, β_2, β_3; γ_1, γ_2, γ_3.

If the point O be fixed, we have at any instant

$$x = a_1\xi + b_1\eta + c_1\zeta, \quad y = a_2\xi + b_2\eta + c_2\zeta, \quad z = a_3\xi + b_3\eta + c_3\zeta.$$

If at this instant any other point of the body besides O occupy the same position in space as at the beginning of the

motion, for this point, $x = \xi$, $y = \eta$, $z = \zeta$, and therefore we should have

$$\begin{vmatrix} a_1 - 1 & b_1 & c_1 \\ a_2 & b_2 - 1 & c_2 \\ a_3 & b_3 & c_3 - 1 \end{vmatrix} = 0.$$

But it is easy to see that this condition is fulfilled, for putting

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \Delta,$$

and denoting the former determinant by Δ', we have, by the multiplication of the determinants,

$$\Delta\Delta' = -\Delta', \text{ and therefore } \Delta' = 0.$$

Hence we conclude that if a rigid body have a fixed point, any two positions have a line in common.

Again,

$$dx = \xi\, da_1 + \eta\, db_1 + \zeta\, dc_1, \quad dy = \xi\, da_2 + \eta\, db_2 + \zeta\, dc_2,$$

$$dz = \xi\, da_3 + \eta\, db_3 + \zeta\, dc_3;$$

but since, at the beginning of the motion, ξ, η, ζ coincide with x, y, z, we have at that instant

$$a_1 = \cos \alpha_1; \; \therefore \; da_1 = -\sin \alpha_1 d\alpha_1 = 0, \text{ since } \alpha_1 = 0.$$

In like manner $\qquad db_2 = 0, \quad dc_3 = 0;$

also $\qquad a_1b_1 + a_2b_2 + a_3b_3 = 0.$

Differentiating, and remembering that initially

$$a_1 = 1, \quad a_2 = 0, \quad a_3 = 0, \quad b_1 = 0, \quad b_2 = 1, \quad b_3 = 0,$$

we have $\qquad db_1 + da_2 = 0.$

In like manner $\quad dc_1 + da_3 = 0, \quad db_3 + dc_2 = 0.$

Let now $da_2 = d\psi, \quad db_3 = d\theta, \quad dc_1 = d\phi$;

then
$$\left.\begin{aligned} dx &= -\eta\, d\psi + \zeta\, d\phi = -y\, d\psi + z\, d\phi \\ dy &= -\zeta\, d\theta + \xi\, d\psi = -z\, d\theta + x\, d\psi \\ dz &= -\xi\, d\phi + \eta\, d\theta = -x\, d\phi + y\, d\theta \end{aligned}\right\}. \qquad (1)$$

But a rotation $d\theta$ round x would give (Art. 222)
$$dy = -\zeta\, d\theta, \quad dz = \eta\, d\theta;$$

$d\phi$ round y would give
$$dz = -\xi\, d\phi, \quad dx = \zeta\, d\phi;$$

and $d\psi$ round z would give
$$dx = -\eta\, d\psi, \quad dy = \xi\, d\psi.$$

Hence the most general infinitely small displacement the body can take, O remaining fixed, is equivalent to rotations round any three rectangular axes through O.

Moreover, from the values of dx, dy, dz, given above, it appears that for a point whose coordinates fulfil the conditions $\frac{\xi}{d\theta} = \frac{\eta}{d\phi} = \frac{\zeta}{d\psi}$ the displacements are zero.

Hence the three rotations $d\theta$, $d\phi$, $d\psi$, round the axis x, y, z, are equivalent to a single rotation round an axis whose position is defined by these equations. If we put
$$d\theta = d\chi \cos\lambda, \quad d\phi = d\chi \cos\mu, \quad d\psi = d\chi \cos\nu,$$

where
$$d\chi = \sqrt{d\theta^2 + d\phi^2 + d\psi^2},$$

the equations of the fixed axis are
$$\frac{\xi}{\cos\lambda} = \frac{\eta}{\cos\mu} = \frac{\zeta}{\cos\nu}.$$

Also, for any point of the body,
$$dx^2 + dy^2 + dz^2 = \left[(\eta\cos\nu - \zeta\cos\mu)^2 + (\zeta\cos\lambda - \xi\cos\nu)^2 + (\xi\cos\mu - \eta\cos\lambda)^2\right] d\chi^2 = p^2\, d\chi^2,$$

if p be the perpendicular from the point on the fixed axis. Hence $d\chi$ is the magnitude of the resultant rotation.

Whence infinitely small rotations, and therefore velocities of rotation, are compounded like forces meeting at a point.

254. **Motion of a Body entirely Free.**—If the point of intersection of the axes fixed in the body be itself in motion, and if its coordinates, referred to axes fixed in space, be x', y', z'; then, for any point xyz of the body,

$$x = x' + a_1\xi + b_1\eta + c_1\zeta, \quad y = y' + a_2\xi + b_2\eta + c_2\zeta,$$
$$z = z' + a_3\xi + b_3\eta + c_3\zeta;$$

whence

$$dx = dx' + \xi\, da_1 + \eta\, db_1 + \zeta\, dc_1, \quad dy = dy' + \xi\, da_2 + \eta\, db_2 + \zeta\, dc_2,$$
$$dz = dz' + \xi\, da_3 + \eta\, db_3 + \zeta dc_3.$$

If we suppose the axes of ξ, η, ζ parallel to those of x, y, z at the beginning of the motion, we get, as in the last Article,

$$\left.\begin{aligned} dx &= dx' - \eta d\psi + \zeta d\phi = dx' - (y - y')\, d\psi + (z - z')\, d\phi \\ dy &= dy' - \zeta\, d\theta + \xi d\psi = dy' - (z - z')\, d\theta + (x - x')\, d\psi \\ dz &= dz' - \xi\, d\phi + \eta\, d\theta = dz' - (x - x')\, d\phi + (y - y')\, d\theta \end{aligned}\right\}, \quad (2)$$

and we see that—

The most general infinitely small displacement which a rigid body can receive consists of a movement of translation, and a movement of rotation round an axis through any arbitrary point of the body.

Again, *whatever be the point through which the axis of rotation is supposed to pass, the direction and magnitude of the rotation remain unaltered.*

Suppose two points $x'y'z'$, $x''y''z''$, successively regarded as the points through which the axis of rotation passes; then,

$$dx = dx'' - (y - y'')\, d\psi'' + (z - z'')\, d\phi'';$$

also
$$dx' = dx'' - (y' - y'')\, d\psi'' + (z' - z'')\, d\phi''.$$

Subtracting, we get

$$dx = dx' - (y - y')\, d\psi'' + (z - z')\, d\phi'';$$

but again,

$$dx = dx' - (y - y')\,d\psi' + (z - z')\,d\phi'.$$

Comparing these, we see that

$$d\psi' = d\psi'', \quad d\phi' = d\phi''.$$

In like manner $d\theta' = d\theta''$; hence the rotation remains unaltered in magnitude and direction.

255. **Velocity of any Point of a Body.**—Infinitely small displacements divided by the element of time during which they are effected become velocities. If the axes of x, y, z be three rectangular axes fixed in space, and if the velocities of rotation round parallel axes meeting at the point $x'y'z'$, be ω_x, ω_y, ω_z, we have, from equations (2),

$$\left.\begin{aligned}
\frac{dx}{dt} &= \frac{dx'}{dt} - (y - y')\,\omega_z + (z - z')\,\omega_y \\
\frac{dy}{dt} &= \frac{dy'}{dt} - (z - z')\,\omega_x + (x - x')\,\omega_z \\
\frac{dz}{dt} &= \frac{dz'}{dt} - (x - x')\,\omega_y + (y - y')\,\omega_x
\end{aligned}\right\}. \qquad (3)$$

If the point $x'y'z'$ be fixed in space, and be taken for the origin, we have

$$\left.\begin{aligned}
\frac{dx}{dt} &= \omega_y z - \omega_z y \\
\frac{dy}{dt} &= \omega_z x - \omega_x z \\
\frac{dz}{dt} &= \omega_x y - \omega_y x
\end{aligned}\right\}. \qquad (4)$$

If we suppose the axes fixed in space to coincide at the instant under consideration with axes fixed in the body, and if the angular velocities round the latter be ω_1, ω_2, ω_3, we have $\omega_x = \omega_1$, $\omega_y = \omega_2$, $\omega_z = \omega_3$. Consequently, if ξ, η, ζ be the coordinates of any point, referred to axes fixed in the body,

and if u, v, w be the components of its velocity parallel to these axes, we have

$$\left.\begin{aligned} u &= \omega_2\zeta - \omega_3\eta \\ v &= \omega_3\xi - \omega_1\zeta \\ w &= \omega_1\eta - \omega_2\xi \end{aligned}\right\}. \tag{5}$$

Equations (3), (4), and (5) hold good for every instant, whereas the equations $x = \xi$, &c., $\omega_x = \omega_1$, &c., $\frac{dx}{dt} = u$, &c., hold good only for one particular instant.

If λ, μ, ν be the direction cosines of a definite line in the body referred to axes parallel to fixed directions in space, we have, as an immediate consequence of (4),

$$\left.\begin{aligned} \frac{d\lambda}{dt} &= \omega_y\nu - \omega_z\mu \\ \frac{d\mu}{dt} &= \omega_z\lambda - \omega_x\nu \\ \frac{d\nu}{dt} &= \omega_x\mu - \omega_y\lambda \end{aligned}\right\}. \tag{6}$$

The motion of a body relative to the space in which it is moving is unaltered if we attribute to the latter the motion of the body reversed, and suppose the body itself to be at rest. Hence, if l, m, n be the direction cosines of a line fixed in space referred to body axes, we may regard the latter as fixed in space, and the line lmn as moving round them with angular velocities $-\omega_1$, $-\omega_2$, $-\omega_3$. Accordingly, from (6), we have

$$\left.\begin{aligned} \frac{dl}{dt} &= -\omega_2 n + \omega_3 m \\ \frac{dm}{dt} &= -\omega_3 l + \omega_1 n \\ \frac{dn}{dt} &= -\omega_1 m + \omega_2 l \end{aligned}\right\}. \tag{7}$$

256. **Acceleration of Rotation.**—If ω_1, ω_2, ω_3, be the angular velocities round three rectangular axes, OA, OB, OC fixed in the body, and ω_x, ω_y, ω_z the velocities round axes OX, OY, OZ fixed in space; and if at any instant we suppose OX, OY, OZ to coincide with the positions occupied at the instant by OA, OB, OC, then not only is ω_1 equal to ω_x, ω_2 to ω_y, and ω_3 to ω_z, but also

$$\frac{d\omega_1}{dt} = \frac{d\omega_x}{dt}, \quad \frac{d\omega_2}{dt} = \frac{d\omega_y}{dt}, \quad \frac{d\omega_3}{dt} = \frac{d\omega_z}{dt}.$$

This may be proved as follows:—

Let ω be the velocity of rotation round a line fixed in the body, which passes through O, and makes angles with the axes OX, OY, OZ, whose direction cosines are λ, μ, ν; then

$$\omega = \omega_x\lambda + \omega_y\mu + \omega_z\nu;$$

therefore

$$\frac{d\omega}{dt} = \frac{d\omega_x}{dt}\lambda + \frac{d\omega_y}{dt}\mu + \frac{d\omega_z}{dt}\nu$$

$$+ \omega_x\frac{d\lambda}{dt} + \omega_y\frac{d\mu}{dt} + \omega_z\frac{d\nu}{dt}.$$

Hence, by (6),

$$\frac{d\omega}{dt} = \lambda\frac{d\omega_x}{dt} + \mu\frac{d\omega_y}{dt} + \nu\frac{d\omega_z}{dt}. \tag{8}$$

This equation shows that the acceleration of rotation round a line is the differential coefficient, with respect to the time, of the angular velocity round the same line even though it is in motion, provided it be fixed in the body.

Thus, in particular, we have in the case supposed above,

$$\frac{d\omega_1}{dt} = \frac{d\omega_x}{dt}, \quad \frac{d\omega_2}{dt} = \frac{d\omega_y}{dt}, \quad \frac{d\omega_3}{dt} = \frac{d\omega_z}{dt}. \tag{9}$$

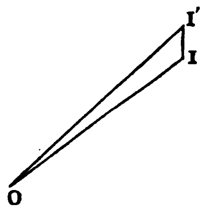
The same may be proved geometrically as follows:—

The body at any instant is rotating round a certain axis with an angular velocity ω. Draw a line through the fixed origin in the direction of the instantaneous axis, and measure off on it a portion OI, proportional to ω; then the projec-

tions of this line on the axes fixed in space represent ω_x, ω_y, ω_z; and its projections on the axes fixed in the body represent ω_1, ω_2, ω_3. At the next instant the body is rotating round another line with a velocity ω', represented by OI', and the projections of OI' represent ω_x', ω_y', ω_z'; ω_1', ω_2', ω_3'. But the projection of OI' is equal to the sum of the projections of OI and II'. Hence

$$d\omega_x = \omega_x' - \omega_x = \text{projection of } II' \text{ on axis of } x \text{ fixed in space,}$$

$$d\omega_1 = \omega_1' - \omega_1 = \text{projection of } II' \text{ on axis of } \xi \text{ fixed in the body.}$$



At the first instant the axes of x and ξ coincide; and at the next the two projections of II' differ only by a quantity infinitely small compared with II', which is itself infinitely small of the first order. Hence $d\omega_x$ and $d\omega_1$ differ by an infinitely small quantity of the second order;

or
$$\frac{d\omega_1}{dt} = \frac{d\omega_x}{dt}, \quad \frac{d\omega_2}{dt} = \frac{d\omega_y}{dt}, \quad \frac{d\omega_3}{dt} = \frac{d\omega_z}{dt}.$$

A line passing through O parallel to II' is called *the axis of angular acceleration*. If we put $\frac{d\omega_x}{dt} = \dot{\omega}_x$, &c., the magnitude of the resultant angular acceleration is $\sqrt{(\dot{\omega}_x^2 + \dot{\omega}_y^2 + \dot{\omega}_z^2)}$, as it is the resultant of the three accelerations $\dot{\omega}_x$, $\dot{\omega}_y$, and $\dot{\omega}_z$.

257. **Accelerations of a Point, parallel to three Axes fixed in the Body.**—If u, v, w be the velocities of a point parallel to axes fixed in the body, its velocity-component V, along a line whose direction cosines referred to these axes are l, m, n, is $ul + vm + wn$.

If we suppose this latter line fixed in space, the acceleration of the point parallel to it is $\frac{dV}{dt}$, and we have

$$\frac{dV}{dt} = l\frac{du}{dt} + m\frac{dv}{dt} + n\frac{dw}{dt} + u\frac{dl}{dt} + v\frac{dm}{dt} + w\frac{dn}{dt}.$$

Substituting the values of $\frac{dl}{dt}$, $\frac{dm}{dt}$, and $\frac{dn}{dt}$, given by (7), we obtain $\frac{dV}{dt}$

$$= l\left(\frac{du}{dt} - v\omega_3 + w\omega_2\right) + m\left(\frac{dv}{dt} - w\omega_1 + u\omega_3\right) + n\left(\frac{dw}{dt} - u\omega_2 + v\omega_1\right).$$

Let us now suppose OX to be the fixed line, then

$l = 1$, $m = n = 0$, and therefore $\frac{dV}{dt} = \frac{du}{dt} - v\omega_3 + w\omega_2$; but $\frac{dV}{dt}$ is now the acceleration of the point parallel to one of the axes fixed in the body; hence we have, for the accelerations of a point parallel to three rectangular axes fixed in the body, the expressions

$$\frac{du}{dt} - v\omega_3 + w\omega_2, \quad \frac{dv}{dt} - w\omega_1 + u\omega_3, \quad \frac{dw}{dt} - u\omega_2 + v\omega_1,$$

where u, v, w are the velocities of the point parallel to the axes fixed in the body.

258. **Complete Determination of the Motion of a Body.**—Every motion which a rigid body can take is reducible to a motion of translation and a motion of rotation. In order then to determine the motion of the body, a point in it is selected (usually the centre of inertia), and the motion of the body is reduced to the motion of this point, together with the rotatory motion of the body round it.

Geometrically the motion may be represented by the rolling of a cone, fixed in the body, on a cone unattached to the body, except at one point (the common vertex of the cones), the latter cone undergoing a motion of translation. If the two cones and the rate at which the one rolls on the other are known, as well as the position in the body of their common vertex, its velocity at each instant, and the path which it describes, then the motion of the body is completely determined.

It is usually most convenient to consider the motion of translation and the motion of rotation separately. The investigation of the former motion is, as we have seen (Art. 205), reducible to the problem of the motion of a particle. The latter motion is completely determined if we can assign at each instant the position of the body and its velocities of rotation in reference to axes, through the centre of inertia, whose directions are fixed in space.

The equations of Kinetics usually give the velocities of rotation round axes *fixed in the body;* but in order fully to determine the motion, it is necessary to ascertain the effect of these velocities when the position of the body is referred to axes whose directions are *fixed in space.* As the points of intersection of these two sets of axes coincide, the velocities of rotation have no effect on the motion of this point O; and therefore, so far as the angular velocities are concerned, we may regard O as fixed, not only in the body, but also in space.

Call the space-axes OX, OY, OZ; the body-axes OA, OB, OC, each set being rectangular.

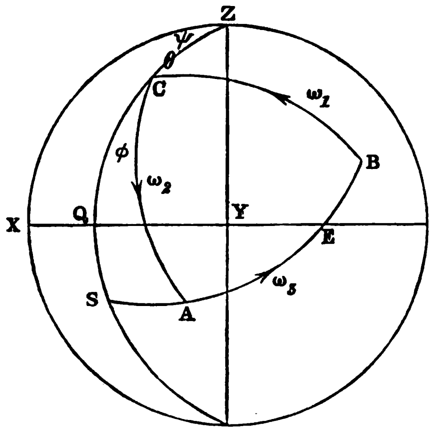
Round the point O as centre describe a sphere, and let the axes meet it at the points X, Y, Z, A, B, C.

Three independent angles are required to determine the position of the body in space.

Those which are probably the best adapted for the solution of the problem are the angular coordinates of the point C, or of the line OC, and the angle ϕ, which the plane COA makes with the plane ZOC. It is obvious that the position of OC fixes the plane AOB, but does not determine the position of the lines OA and OB in this plane. Hence, when C is fixed, if the angle ϕ which the plane COA makes with the plane ZOC be given, the position of the body is completely determined. The angular coordinates of OC are θ, the angle which it makes with OZ, and ψ, the angle which the plane COZ makes with the plane XOZ.

Suppose now that the body has three velocities of rota-

tion: ω_1, round OA; ω_2 round OB; and ω_3 round OC, in the direction of the arrow heads. We have to express $\frac{d\theta}{dt}$, $\frac{d\phi}{dt}$, and $\frac{d\psi}{dt}$ in terms of these velocities, remembering that the changes of θ, ϕ, and ψ are caused solely by ω_1, ω_2, ω_3.



The motion of the point C on the sphere is unaffected by ω_3. If the radius of the sphere be unity, the point C has two velocities, ω_1 and ω_2, along the tangents to the great circles BC and CA. Resolving these velocities along the great circle ZC, and at right angles to it, we have

$$\frac{d\theta}{dt} = \omega_2 \cos\phi + \omega_1 \sin\phi, \tag{10}$$

$$\sin\theta \frac{d\psi}{dt} = \omega_2 \sin\phi - \omega_1 \cos\phi. \tag{11}$$

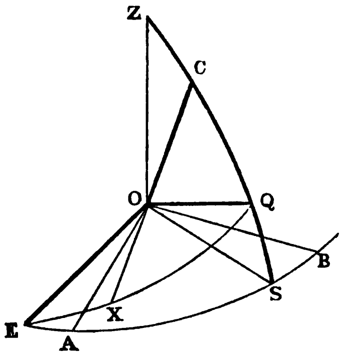
These equations are obvious, since the arc of a small circle, on a sphere whose radius is unity, is equal to the angle subtended at its pole, multiplied by the sine of the spherical radius. As regards the algebraical signs it is well to observe that ψ is counted from ZX towards ZY; and that ϕ is

positive and acute, when E lies in the quadrant AB.

The value of $\frac{d\phi}{dt}$ is easily obtained by considering the motion of the point A in the plane AB. The angular velocity of A in the plane AB is ω_3, but it is also the component along the great circle AB of the velocity of S together with the rate of increase of SA, i.e. of ϕ; and since SZ and SA are at right angles, and S lies on ZC at a distance $90° + \theta$ from Z, the velocity of S along AB is $\cos\theta \frac{d\psi}{dt}$, therefore

$$\omega_3 = \cos\theta \frac{d\psi}{dt} + \frac{d\phi}{dt}, \text{ whence } \frac{d\phi}{dt} = \omega_3 - \cos\theta \frac{d\psi}{dt}. \quad (12)$$

The angles made use of by Laplace in his solution of the problem of Precession and Nutation are somewhat different from those considered above. Laplace supposes that a point which is moving from X to Y approaches nearer to C after passing E, and he further places E behind A and X. In this way the various lines and planes assume the positions represented in the accompanying diagram.



The angles employed by Laplace are ZOC, which we may denote by θ', EOA, or ϕ', and XOE, or ψ', the last being positive when E is behind X. Taking into account the mode in which Laplace supposes the axes to be situated, we have,

then, $\theta' = -\theta$, $\phi' = \phi - \frac{1}{2}\pi$, $\psi' = \frac{1}{2}\pi - \psi$, and equations (10), (11), and (12) become, by substitution,

$$\left.\begin{aligned} \frac{d\theta'}{dt} &= \omega_2 \sin\phi' - \omega_1 \cos\phi' \\ \sin\theta' \frac{d\psi'}{dt} &= \omega_1 \sin\phi' + \omega_2 \cos\phi' \\ \frac{d\phi'}{dt} &= \omega_3 + \cos\theta' \frac{d\psi'}{dt} \end{aligned}\right\} . \qquad (13)$$

Omit

259. **Screws and Twists.**—It was shown in Article 252 that a body can be moved from any one position into any other, by a translation combined with a rotation, round an axis through any arbitrary point of the body.

The translation may be resolved into two—one parallel to the axis of rotation, and the other at right angles thereto. The latter translation, along with the rotation, may be replaced by a pure rotation round a parallel axis, and so the whole motion will consist of a translation parallel to a certain fixed line and of a rotation round it. Such a motion is similar to that of a nut on a screw, and is called a *Twist*. Hence a body can be moved from any one position into any other by means of a twist.

In order to determine a screw it is necessary to specify—(1) the position and direction of the line round which the rotation is effected, or *the axis of the screw;* and (2) the ratio of the translation to the rotation. This last is a linear magnitude, and is called *the pitch of the screw*. In order to determine a twist, we must, in addition to the screw round which it is effected, specify its amplitude, *i. e.* the magnitude of the rotation.

The twist by which a body can be moved from any one position into any other is in general unique.

This readily appears from considering that if two positions of a body are given, the magnitude of the corresponding rotation and the direction of its axis are invariable; and that if two positions of a plane figure in its own plane are

given, the position of the corresponding centre of rotation is thereby determined.

The same thing is proved directly by Sir Robert Ball (to whom the *Theory of Screws* is principally due), as follows:—

Any point of the body, which lies on the axis of the twist, must continue thereon after the motion. If, therefore, the motion could be effected by two different twists, there would be two different lines along which points of the body would continue throughout the motion. In order that this should be possible, the lines must be parallel, and the motion one of pure translation.

If two successive positions of a body in motion are infinitely near each other, the twist by which it can be brought from the one position to the other is the actual motion of the body. We see then that the most general motion of a rigid body consists of a succession of twists. The screw round which it is twisting at any instant is called the *instantaneous screw*. As the position of a straight line in space is determined by four independent quantities, five magnitudes must be assigned to determine a screw. In order to determine a twist, its amplitude, and the pitch, as well as the position of the axis, of the corresponding screw, are required. Hence the motion of a rigid body in general depends on six independent variables, and we see, as in Article 215, that a rigid body entirely unrestrained has six degrees of freedom.

260. **Composition of Twists.**—If a body receive in succession two twists whose amplitudes are infinitely small, the order in which they are effected is indifferent, and the resulting change of position may be produced by a single twist, which is the resultant of the two former.

More symmetrical results are obtained, if instead of seeking for the twist which is the resultant of two others, we inquire how three twists having infinitely small amplitudes must be related, in order that the position of a body, after being affected by them, may remain unaltered.

The question proposed may be solved directly, but the method of solution devised by Sir Robert Ball leads to results of a more instructive character. This mode of solution will be found in Example 14.

EXAMPLES.

1. Determine the velocity with which the plane of the horizon, at a place whose latitude is given, turns round a vertical axis.

Ans. $\omega \sin \lambda$, where ω is the earth's angular velocity, and λ the latitude.

2. If the velocities of rotation of a body round three rectangular axes are given in terms of the time, show how to determine—(1) the velocity of rotation round the instantaneous axis; (2) the position of the instantaneous axis; (3) the equation of the cone which is the locus of the instantaneous axis.

3. If the velocities of rotation round three rectangular axes are proportional to the time which has elapsed from a given epoch, the position of the instantaneous axis is fixed.

4. If the accelerations of rotation round three rectangular axes are constant, the instantaneous axis lies in a fixed plane.

5. If θ, ϕ, ψ, ω_1, ω_2, ω_3 have the same significations as in Art. 258, show that

$$\omega_1 = \sin\phi \frac{d\theta}{dt} - \sin\theta\cos\phi\frac{d\psi}{dt},$$

$$\omega_2 = \cos\phi \frac{d\theta}{dt} + \sin\theta\sin\phi\frac{d\psi}{dt},$$

$$\omega_3 = \frac{d\phi}{dt} + \cos\theta\frac{d\psi}{dt}.$$

6. A body is rotating round a fixed point O. If OX, OY, OZ be rectangular axes fixed in space, and OA, OB, OC rectangular axes fixed in the body; and if the direction cosines of the latter referred to the former be, respectively, a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3; show that

$$\frac{da_1}{dt} = b_1\omega_3 - c_1\omega_2, \quad \frac{db_1}{dt} = c_1\omega_1 - a_1\omega_3, \quad \frac{dc_1}{dt} = a_1\omega_2 - b_1\omega_1,$$

$$\frac{da_2}{dt} = b_2\omega_3 - c_2\omega_2, \quad \frac{db_2}{dt} = c_2\omega_1 - a_2\omega_3, \quad \frac{dc_2}{dt} = a_2\omega_2 - b_2\omega_1,$$

$$\frac{da_3}{dt} = b_3\omega_3 - c_3\omega_2, \quad \frac{db_3}{dt} = c_3\omega_1 - a_3\omega_3, \quad \frac{dc_3}{dt} = a_3\omega_2 - b_3\omega_1,$$

where ω_1, ω_2, ω_3 are the angular velocities of the body round OA, OB, OC.

7. Deduce equations (10), (11), (12), Art. 258, from equations (7), Art. 255.

8. A body receives in a given order rotations of finite magnitude round two axes fixed in space, or in the body, and meeting in a point. Find the position of the axis, a single rotation round which would bring the body into the same position, and determine the magnitude of the resultant rotation.

This question is solved in a manner similar to that employed in Examples 3 and 4, Art. 226; the construction in the present case being on the surface of a sphere instead of a plane.

When the rotations round the given axes are in the same direction, the resultant rotation is double the supplement of the vertical angle of a spherical triangle, whose base and base angles are the angle between the axes and the semi-amplitudes of the rotations round them.

9. A rigid body receives a motion of translation, whose components, parallel to the axes, are a, b, c, and a rotation θ round an axis fixed in the body, which, at the beginning of the motion, coincides with the axis of z. Determine the position and pitch of the screw, a twist round which would bring the body into the same position; and find the amplitude of the twist.

The screw passes through a point whose coordinates are

$$x = \frac{a \sin \frac{1}{2}\theta - b \cos \frac{1}{2}\theta}{2 \sin \frac{1}{2}\theta}, \quad y = \frac{b \sin \frac{1}{2}\theta + a \cos \frac{1}{2}\theta}{2 \sin \frac{1}{2}\theta}.$$

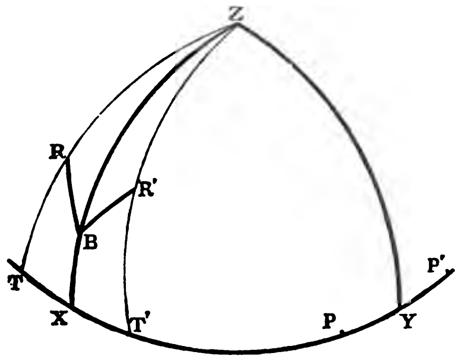
Pitch of screw $= \frac{c}{\theta}$. Amplitude of twist $= \theta$.

10. A body receives, in succession, rotations of finite magnitude round two non-intersecting axes a, b, either fixed in space or fixed in the body: if d be the shortest distance between the lines a and b; θ and θ' the amplitudes of the rotations round them; ϵ the angle between them; ϕ the amplitude of the twist equivalent to the motion; and p the pitch of its screw; prove that

$$\tfrac{1}{2}p\phi \sin \tfrac{1}{2}\phi = d \sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta' \sin \epsilon.$$

(This theorem is due to Rodrigues: *Liouville*, T. 5, p. 390.)

Take the shortest distance between a and b for axis of y; the point of intersection of this line with b for origin O; and a parallel to a for axis of z.



After the body has received its rotation round a, suppose it receives in succession two equal and opposite rotations round OZ, the first of these being equal and opposite to that round a. These rotations, being equal and opposite, do not change the position of the body.

First, suppose a and b to be fixed in space, then so also is OZ.

The rotation round a and the equal and opposite one round OZ are (Ex. 5, Art. 226) equivalent to a translation, whose magnitude is $2d \sin \frac{1}{2}\theta$, and whose direction lies in the plane XOY, and is at right angles to a line OP which makes with OY an angle $-\frac{1}{2}\theta$.

Describe a sphere round O as centre, and let B be the point in which it is met by b, then $ZB = \epsilon$. The axis of the rotation, which is equivalent (Ex. 8) to the rotations round OZ and b, meets the sphere in R, and the direction of translation meets it in T; where

$$TX = \tfrac{1}{2}\theta, \quad RZX = \tfrac{1}{2}\theta, \quad ZBR = \tfrac{1}{2}\theta'.$$

Then, by Ex. 8,

$$TRB = \tfrac{1}{2}\phi,$$

and we have $p\phi$ = component of translation parallel to axis of screw

$$= 2d \sin \tfrac{1}{2}\theta \cos TR = 2d \sin \tfrac{1}{2}\theta \sin ZR,$$

whence

$$\tfrac{1}{2}p\phi \sin \tfrac{1}{2}\phi = d \sin \tfrac{1}{2}\theta \sin ZR \sin \tfrac{1}{2}\phi = d \sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}\theta' \sin \epsilon.$$

Secondly, if a and b be fixed in the body, so, likewise, is OZ. In this case the line OP becomes OP', which makes an angle $\frac{1}{2}\theta$ with OY, and the points T and R become T' and R', where $XT' = \frac{1}{2}\theta$, $R'BZ = \frac{1}{2}\theta'$. Then (by Ex. 8) $BR'T' = \frac{1}{2}\phi$, and the result is obtained in the same manner as before.

11. A body receives twists, having infinitely small amplitudes, round two screws intersecting at right angles. Determine the amplitude of the resultant twist, and the position and pitch of its screw.

Take, for the axes of x and y, the axes of the screws; let their pitches be p and q, and the amplitudes of the twists round them θ and ϕ.

The rotations θ and ϕ are equivalent to a single rotation ψ round an axis lying in the plane xy, and making an angle λ with the axis of x, where

$$\psi \cos \lambda = \theta, \quad \psi \sin \lambda = \phi.$$

The translations $p\theta$ and $q\phi$ are equivalent to a translation

$$p\theta \cos \lambda + q\phi \sin \lambda = (p \cos^2 \lambda + q \sin^2 \lambda)\,\psi$$

along the axis of the resultant rotation, and to a translation

$$q\phi \cos \lambda - p\theta \sin \lambda = (q - p)\,\psi \sin \lambda \cos \lambda$$

at right angles thereto. The latter translation, together with the rotation ψ round the axis through the origin, is equivalent to a rotation ψ round a parallel axis passing through a point on the axis of z, whose distance from the origin is

$$(q - p) \sin \lambda \cos \lambda.$$

Hence the position of the axis of the screw corresponding to the resultant twist is given by the equations

$$y = x \tan \lambda, \quad z = (q - p) \sin \lambda \cos \lambda,$$

where
$$\tan\lambda = \frac{\phi}{\theta};$$

and its pitch by the equation $r = p\cos^2\lambda + q\sin^2\lambda$.

Also, if ψ be the amplitude of the resultant twist, we have

$$\psi^2 = \theta^2 + \phi^2.$$

12. Any screw, a twist round which is the resultant of twists round two given screws intersecting at right angles, lies on a surface determined by the given screws; and its pitch depends on the angle which its direction makes with one of these screws. The amplitudes of the twists are supposed to be infinitely small.

If we eliminate λ from the equations of the last example, which define the position of the screw belonging to the resultant twist, we obtain

$$z(x^2 + y^2) - (q - p)\,xy = 0,$$

which is the equation of the surface.

This surface is called the *cylindroid* by Sir Robert Ball.

The pitch r of the screw is, as shown in the last example, given by the equation
$$r = p\cos^2\lambda + q\sin^2\lambda.$$

If we describe in the plane of xy the conic whose equation is

$$px^2 + qy^2 = \epsilon^3,$$

where ϵ is a linear constant, the square of the reciprocal of any diameter of this conic is proportional to the pitch of the parallel screw on the cylindroid. When a screw is spoken of as belonging to the cylindroid, it is understood that not only is its axis one of the generating lines of the surface, but also that its pitch is defined in the manner just mentioned.

13. Prove that any two screws belong to the same cylindroid. Also two screws being given, determine the cylindroid to which they belong.

Take the common perpendicular to the two screws as axis of z; we have then to determine the position of the origin and of the axis of x, and the magnitudes of the quantities p and q, so as to satisfy the equations

$$r_1 = p\cos^2\lambda_1 + q\sin^2\lambda_1, \qquad z_1 = (q - p)\sin\lambda_1\cos\lambda_1,$$

$$r_2 = p\cos^2\lambda_2 + q\sin^2\lambda_2, \qquad z_2 = (q - p)\sin\lambda_2\cos\lambda_2,$$

$$A = \lambda_1 - \lambda_2, \qquad h = z_1 - z_2,$$

where A is the angle, and h the distance between the given screws. As the number of quantities at our disposal is equal to the number of equations to be satisfied, it is always possible to determine a cylindroid containing the given screws. The equations are solved as follows:—Subtracting,

$$r_1 - r_2 = (q - p)(\sin^2\lambda_1 - \sin^2\lambda_2) = (q - p)\sin(\lambda_1 + \lambda_2)\sin(\lambda_1 - \lambda_2),$$
$$h = (q - p)\cos(\lambda_1 + \lambda_2)\sin(\lambda_1 - \lambda_2).$$

Adding,

$$r_1 + r_2 = 2p - (p - q)(\sin^2\lambda_1 + \sin^2\lambda_2)$$

$$= p + q + \tfrac{1}{2}(p - q)(\cos 2\lambda_1 + \cos 2\lambda_2)$$

$$= p + q + (p - q)\cos(\lambda_1 + \lambda_2)\cos(\lambda_1 - \lambda_2),$$

$$z_1 + z_2 = (q - p)\sin(\lambda_1 + \lambda_2)\cos(\lambda_1 - \lambda_2);$$

hence $$(q - p)\sin(\lambda_1 + \lambda_2) = \frac{r_1 - r_2}{\sin A}, \quad z_1 + z_2 = (r_1 - r_2)\cot A,$$

$$(q - p)\cos(\lambda_1 + \lambda_2) = \frac{h}{\sin A}, \quad z_1 - z_2 = h,$$

$$p + q = r_1 + r_2 + h\cot A, \quad \lambda_1 - \lambda_2 = A,$$

and the mode of completing the solution is obvious.

14. A body receives three twists having infinitely small amplitudes. Determine the relations between the twists, in order that the position of the body should remain unaltered.

Sir Robert Ball's solution is as follows:—

Determine the cylindroid containing two of the screws. Take its screws intersecting at right angles for axes of x and y. Let Ω_1, Ω_2, Ω_3 be the amplitudes of the three twists, and λ_1, λ_2, λ_3 the angles which their screws make with the axis of x. If the third screw belongs to the cylindroid containing the other two, and if the angle it makes with the axis of x and the amplitude of the corresponding twist satisfy the equations

$$\frac{\Omega_1}{\sin(\lambda_2 - \lambda_3)} = \frac{\Omega_2}{\sin(\lambda_3 - \lambda_1)} = \frac{\Omega_3}{\sin(\lambda_1 - \lambda_2)},$$

the twists compensate each other.

In fact each twist can be resolved into two round the screws lying along the axes of x and y. The whole motion is thus reduced to two twists round these screws; and if the amplitudes of these twists are zero, the body remains undisturbed. But the equations above are the conditions that the rotations round the axes of x and y should be zero, and these rotations are the amplitudes of the twists.

As the twist by which a given motion can be effected is unique, there is only one twist by which two given twists can be compensated; and, therefore, if three twists compensate each other, the third screw must belong to the cylindroid containing the other two, and the above equations must hold good.

15. A body is moving round a fixed point. Determine the accelerations of any point parallel and at right angles to the instantaneous axis of rotation.

Taking three lines fixed in space through the fixed point as axes,

$$\frac{dx}{dt} = \omega_y z - \omega_z y, \quad \frac{dy}{dt} = \omega_z x - \omega_x z, \quad \frac{dz}{dt} = \omega_x y - \omega_y x;$$

differentiating, and substituting for $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, from these equations, we obtain,

$$\frac{d^2 x}{dt^2} = \omega_x(\omega_x x + \omega_y y + \omega_z z) - \omega^2 x + z\frac{d\omega_y}{dt} - y\frac{d\omega_z}{dt},$$

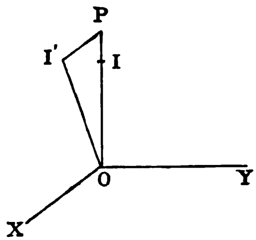
$$\frac{d^2 y}{dt^2} = \omega_y(\omega_x x + \omega_y y + \omega_z z) - \omega^2 y + x\frac{d\omega_z}{dt} - z\frac{d\omega_x}{dt},$$

$$\frac{d^2 z}{dt^2} = \omega_z(\omega_x x + \omega_y y + \omega_z z) - \omega^2 z + y\frac{d\omega_x}{dt} - x\frac{d\omega_y}{dt},$$

remembering that $\qquad \omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2.$

Let us now suppose the axis of z to coincide with OI, the instantaneous axis, then $\omega_x = 0$, $\omega_y = 0$, $\omega_z = \omega$. Let the plane of xz pass through OI', the consecutive position of the instantaneous axis. Measure off OI proportional to ω on OZ, and take OI' proportional to the corresponding angular velocity $\omega + d\omega$; draw $I'P$ perpendicular to OI; then $\omega + d\omega$ round OI' is resolvable into OP round OZ, and $I'P$ round OX.

Let $I'OP = d\psi$; then $I'P = OI'\, d\psi$; therefore



$$\frac{d\omega_x}{dt} = \omega\frac{d\psi}{dt} = \omega\dot{\psi},$$

if the angular velocity of the instantaneous axis be denoted by $\dot{\psi}$. Also

$$\frac{d\omega_y}{dt} = 0, \quad \frac{d\omega_z}{dt} = \frac{d\omega}{dt}, \quad \text{since } d\omega_z \propto IP = OI' - OI.$$

Introducing these, we obtain

$$\frac{d^2 x}{dt^2} = -\omega^2 x - \frac{d\omega}{dt}y, \quad \frac{d^2 y}{dt^2} = \frac{d\omega}{dt}x - \omega^2 y - \omega\dot{\psi}z, \quad \frac{d^2 z}{dt^2} = \omega\dot{\psi}y.$$

16. Find the position of the acceleration-centre in a body rotating round a fixed point.

The only acceleration-centre which in general exists is the fixed point itself.

17. A body is moving round a fixed point O. If perpendiculars, whose lengths are p and q, be let fall from any point A of the body on OI, the instantaneous axis of rotation, and on OJ, that of angular acceleration; prove that the total acceleration of A is the resultant of two components, $\omega^2 p$ along p and σq perpendicular to the plane AOJ, where ω and σ are the resultant angular velocity and angular acceleration of the body.

If x, y, z be the coordinates of A referred to space axes through O, and r be the distance OA, the equation for the acceleration $\ddot{x}$ may be written (Ex. 15),

$$\ddot{x} = \omega^2 \left\{\frac{\omega_x}{\omega}\left(x\frac{\omega_x}{\omega} + y\frac{\omega_y}{\omega} + z\frac{\omega_z}{\omega}\right) - x\right\} + \sigma r\left(\frac{\dot{\omega}_y}{\sigma}\frac{z}{r} - \frac{\dot{\omega}_z}{\sigma}\frac{y}{r}\right),$$

where

$$\sigma^2 = \dot{\omega}_x^2 + \dot{\omega}_y^2 + \dot{\omega}_z^2.$$

If P be the point in which the perpendicular from A meets OI, and if we consider the projection of the triangle OPA on the axis of x, we have projection of AP = projection of OP − projection of OA. From this it is plain that the term by which ω^2 is multiplied in $\ddot{x}$ is the projection of p on the axis of x. Again, if λ, μ, ν be the direction cosines of the normal to the plane AOJ, and θ the angle between OJ and OA, we have

$$\lambda \sin\theta = \frac{\dot{\omega}_y}{\sigma}\frac{z}{r} - \frac{\dot{\omega}_z}{\sigma}\frac{y}{r}, \quad \text{and} \quad r\sin\theta = q;$$

whence it appears that the term by which σ is multiplied in $\ddot{x}$ is $q\lambda$, or the projection of q on the axis of x. The truth of the theorem above is now obvious. This theorem is due to Professor Minchin.

18. A body is rotating round a fixed point: find the locus of a point whose acceleration along its path at any given instant is zero.

As the path at the instant touches a circle, having its centre on the instantaneous axis and its plane at right angles thereto, if p be the distance of any point from the axis,

$$\text{the tangential acceleration} = -\frac{y}{p}\frac{d^2x}{dt^2} + \frac{x}{p}\frac{d^2y}{dt^2} = \frac{x^2+y^2}{p}\frac{d\omega}{dt} - \omega\dot{\psi}\frac{xz}{p}.$$

The required locus is therefore the cone

$$\frac{d\omega}{dt}(x^2+y^2) - \omega\dot{\psi}xz = 0.$$

19. Show that a point whose normal acceleration at right angles to the instantaneous axis vanishes lies on the cone

$$\omega(x^2+y^2) + \dot{\psi}yz = 0.$$

20. A body is rotating round a fixed point: determine at any instant the positions of the osculating plane, and of the principal normal, to the path described by one of its points.

The normal plane to the path is the plane passing through the point and the instantaneous axis. Hence the perpendicular to the osculating plane is the intersection of this plane with its consecutive position. Again, the direction of the principal normal coincides with that of the resultant normal acceleration; hence, if ν be the angle the principal normal makes with the instantaneous axis,

$$\tan\nu = -\frac{\omega p^2 + \dot{\psi}yz}{\dot{\psi}py}.$$

21. Find the radius of curvature of the path of any point of the body.

If N be whole normal acceleration,

$$\rho = \frac{\omega^2 p^2}{N} = \frac{\omega p^3}{\sqrt{\{(\omega p^2 + \dot{\psi} yz)^2 + \dot{\psi}^2 y^2 p^2\}}}.$$

22. A body is moving in any manner. Determine the accelerations of a point parallel and at right angles to the axis of the instantaneous screw.

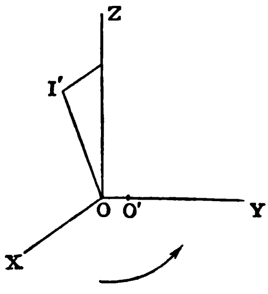
Let x_0, y_0, z_0 be the coordinates of a point fixed in the body, ξ, η, ζ the coordinates of any point referred to x_0, y_0, z_0 as origin; then

$$\frac{d^2x}{dt^2} = \frac{d^2x_0}{dt^2} + \omega_x(\omega_x\xi + \omega_y\eta + \omega_z\zeta) - \omega^2\xi + \zeta\frac{d\omega_y}{dt} - \eta\frac{d\omega_z}{dt},$$

$$\frac{d^2y}{dt^2} = \frac{d^2y_0}{dt^2} + \omega_y(\omega_x\xi + \omega_y\eta + \omega_z\zeta) - \omega^2\eta + \xi\frac{d\omega_z}{dt} - \zeta\frac{d\omega_x}{dt},$$

$$\frac{d^2z}{dt^2} = \frac{d^2z_0}{dt^2} + \omega_z(\omega_x\xi + \omega_y\eta + \omega_z\zeta) - \omega^2\zeta + \eta\frac{d\omega_x}{dt} - \xi\frac{d\omega_y}{dt}.$$

Take as x_0, y_0, z_0 that point of the body which at the instant coincides with the point O on the instantaneous screw in space which is nearest the consecutive position of the instantaneous screw. If C be the ruled surface in space generated by the positions of the instantaneous screw-axis, O will be the point of intersection of the instantaneous screw-axis with the line of striction on C. Let OO' be an element of this line of striction.



At the time $t + dt$ the body is twisting round a screw through O'. Let T be the velocity of translation at the time t, and T' and ω' the velocities of translation and rotation at the time $t + dt$.

Now, the velocity of rotation ω' round $O'S$ (screw-axis through O') is equivalent to ω' round OI' (parallel to $O'S$), and a velocity of translation $\omega'.OO'$ at right angles to OO' and OI'.

The velocity of rotation ω' round OI' is equivalent to ω' round OZ, and $\omega' d\psi$ round OX. Hence, at the time $t + dt$, the point $x_0y_0z_0$ has two velocities of translation: T' along OZ, and $(\omega'.OO' + T'd\psi)$ along OX. Again, as OO' is infinitely small of the first order, the velocity of translation along OZ resulting from $OO'.\omega'$ is infinitely small of the second order. At the time t the point $x_0y_0z_0$ had the velocity T along OZ. Hence, if U be the velocity of translation, and $\dot{\psi}$ the angular velocity of the axis of the instantaneous screw, at the instant, we have

$$\frac{d\psi}{dt} = \dot{\psi}, \qquad \frac{d^2x_0}{dt^2} = \omega U + T\dot{\psi}, \qquad \frac{d^2y_0}{dt^2} = 0, \qquad \frac{d^2z_0}{dt^2} = \frac{dT}{dt}.$$

Also $\omega_x = 0, \quad \omega_y = 0, \quad \omega_z = \omega, \quad \dfrac{d\omega_x}{dt} = \omega\dot{\psi}, \quad \dfrac{d\omega_y}{dt} = 0, \quad \dfrac{d\omega_z}{dt} = \dfrac{d\omega}{dt};$

whence
$$\frac{d^2x}{dt^2} = \omega U + T\dot{\psi} - \omega^2 \xi - \frac{d\omega}{dt}\eta,$$

$$\frac{d^2y}{dt^2} = \frac{d\omega}{dt}\xi - \omega^2 \eta - \omega\dot{\psi}\zeta,$$

$$\frac{d^2z}{dt^2} = \frac{dT}{dt} + \omega\dot{\psi}\eta.$$

23. A body is moving in any way: determine the position of the acceleration-centre at any instant.

Its coordinates are formed from the equations of the last example, by making

$$\frac{d^2x}{dt^2} = 0, \quad \frac{d^2y}{dt^2} = 0, \quad \frac{d^2z}{dt^2} = 0.$$

24. A body is moving in any way; the acceleration at any instant of any point is the same as if the body were rotating round the acceleration-centre as an absolutely fixed point.

Express the accelerations, by the last example, in terms of the coordinates relative to the acceleration-centre as origin, and the results are the same as the expressions of Ex. 15 would become, if we made

$$\omega_x = 0, \quad \omega_y = 0, \quad \omega_z = \omega, \quad \frac{d\omega_x}{dt} = \omega\dot{\psi}, \quad \frac{d\omega_y}{dt} = 0, \quad \frac{d\omega_z}{dt} = \frac{d\omega}{dt}.$$

The theorem is likewise obvious, *à prori*.

The theorems contained in the Examples given above are taken from Schell's *Theorie der Bewegung und der Kräfte*, to which the student is referred for more extended investigations on the subject.

25. Show that the theorem of Ex. 17 holds good for a body moving freely provided the acceleration-centre be substituted for the fixed point O.

This extension of Ex. 17 is due to Professor Minchin.

26. A right circular cone is rolling on another fixed in space, the two cones having a common vertex. Given the velocity of rotation of the rolling cone, determine the velocity with which the plane passing through the instantaneous axis turns round the axis of the fixed cone.

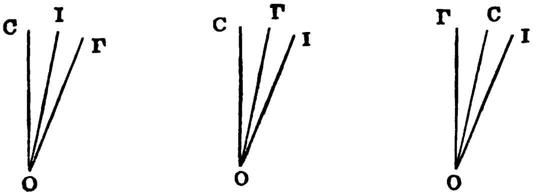
The normal plane through the instantaneous axis contains the axes of both cones. Hence the angle between the two axes remains invariable; and a point on the axis of the rolling cone describes a circle having its centre on the axis of the fixed cone, and its plane perpendicular thereto. It is also at any instant rotating round the instantaneous axis. If we equate the two expressions for the velocity of this point, we get

$$\omega \sin \Gamma = \Omega \sin (C + \Gamma), \tag{1}$$

or
$$-\omega \sin \Gamma = \Omega \sin (C - \Gamma), \tag{2}$$

or
$$\sin \Gamma = \Omega \sin (\Gamma - C), \tag{3}$$

where ω is the angular velocity of the rolling cone round the instantaneous axis; Ω the angular velocity of the plane containing the instantaneous axis round the axis of the fixed cone; C and Γ the semi-angles of the fixed and moving cones.



The first, second, or third formula is to be used, according as—(1) the cones are *outside* one another, having convex surfaces in contact; (2) the rolling cone is a small cone rolling *inside* a larger one; (3) the rolling cone is the larger cone, and rolls outside a smaller fixed cone, which it contains within it.

In each of these figures OC is the axis of the fixed cone; $O\Gamma$ of the rolling cone; and OI the instantaneous axis. If the angles are supposed to contain their signs implicitly, each being measured from OI, the last formula contains the other two.

27. A body is moving round a fixed point. The motion of the instantaneous axis in the body being completely given, determine its motion in space.

Describe a sphere of radius a round the fixed point: the cone C fixed in space and the cone Γ fixed in the body trace out curves on this sphere, and the motion is accomplished by the one curve rolling on the other. The osculating circle of each of these curves, as it passes through three points on the surface of the sphere, will be a circle of the sphere; and the rolling at any instant will be the same as if one of these circles rolled on the other, or as if the right cone on the osculating circle of Γ as base rolled on the right cone, having the osculating circle of C as base. Let r be the radius of curvature of the curve C; ρ of the curve Γ; then

$$\sin C = \frac{r}{a}, \qquad \sin \Gamma = \frac{\rho}{a}, \quad \text{and therefore (Ex. 26),}$$

$$\omega \frac{\rho}{a} = \Omega \sin\left\{\sin^{-1}\frac{\rho}{a} - \sin^{-1}\frac{r}{a}\right\}.$$

Now, if s be the arc of the curve C, and $\frac{ds}{dt}$ the velocity of the point of contact of Γ along it,

$$\Omega = \frac{1}{r}\frac{ds}{dt} = \frac{1}{r}\frac{d\sigma}{dt},$$

where σ is the arc of Γ; whence

$$\omega \frac{\rho}{a} = \frac{1}{r}\frac{d\sigma}{dt} \sin\left\{\sin^{-1}\frac{\rho}{a} - \sin^{-1}\frac{r}{a}\right\}.$$

From this equation r can be determined in terms of t (in terms of which ρ, ω, and $\frac{d\sigma}{dt}$ are supposed to be expressed); and, as $\frac{ds}{dt} = \frac{d\sigma}{dt}$, by eliminating t an equation is obtained between r and s, which is the equation of the curve C; therefore, &c.

SECTION II.—*Kinetics.*

261. **Moments of Momentum of a Body having a Fixed Point.**—If x, y, z be the coordinates of any point of the body, referred to space axes intersecting at the fixed point O; and H_x, H_y, H_z, the moments of momentum round these axes, we have $H_x = \Sigma m (y\dot{z} - z\dot{y})$. Substituting for $\dot{y}$ and $\dot{z}$ their values given by (4), Art. 255, we obtain

$$H_x = \omega_x \int (y^2 + z^2)\, dm - \omega_y \int xy dm - \omega_z \int xz dm.$$

Hence, if a, b, c, i, j, k be the moments and products of inertia of the body at any instant, round the space axes, we have

$$\left.\begin{aligned} H_x &= a\omega_x - k\omega_y - j\omega_z \\ H_y &= -k\omega_x + b\omega_y - i\omega_z \\ H_z &= -j\omega_x - i\omega_y + c\omega_z \end{aligned}\right\}. \qquad (1)$$

If the space axes coincide with the instantaneous position of the principal axes of the body at O, equations (1) become

$$H_1 = A\omega_1, \; H_2 = B\omega_2, \; H_3 = C\omega_3, \qquad (2)$$

where H_1, H_2, H_3, ω_1, ω_2, ω_3 are the moments of momentum and the angular velocities round the principal axes at the instant; and A, B, C are the principal moments of inertia of the body for the point O.

The resultant moment of momentum H is given by the equation

$$H^2 = A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2. \qquad (3)$$

The direction cosines of the momentum axis relative to the principal axes through O are proportional to $A\omega_1$, $B\omega_2$, $C\omega_3$.

If a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 be the direction cosines of the principal axes at O referred to the space axes, we have

$$\left.\begin{aligned} H_x &= A\omega_1 a_1 + B\omega_2 b_1 + C\omega_3 c_1 \\ H_y &= A\omega_1 a_2 + B\omega_2 b_2 + C\omega_3 c_2 \\ H_z &= A\omega_1 a_3 + B\omega_2 b_3 + C\omega_3 c_3 \end{aligned}\right\} . \qquad (4)$$

If O be a definite point of the body not fixed in space, equations (1), (2), (3), (4) still hold good for the motion *relative to O*; the axes x, y, and z being parallel to fixed directions in space.

262. **Motion of a Body having a Fixed Point under the Action of Impulses.**—If a body having a fixed point O be acted on by any set of impulses, whose moments round the principal axes through O are L, M, N; these moments are equal respectively to the changes in the moments of momentum of the body. Hence ((2), Art. 262),

$$A(\omega_1 - \omega_1') = \underset{\cdot}{L}, \; B(\omega_2 - \omega_2') = \underset{\cdot}{M}, \; C(\omega_3 - \omega_3') = \underset{\cdot}{N}, \quad (5)$$

where ω_1', ω_2', ω_3', and ω_1, ω_2, ω_3 are the angular velocities round the principal axes immediately before and immediately after the action of the impulses.

In some cases it may be convenient to use the expressions for H_x, H_y, H_z given in (1), Art. 261, and the moments $\underset{\cdot}{G_x}$, $\underset{\cdot}{G_y}$, $\underset{\cdot}{G_z}$ of the impulses round the space axes. We have, then,

$$\left.\begin{aligned} a(\omega_x - \omega_x') - k(\omega_y - \omega_y') - j(\omega_z - \omega_z') &= \underset{\cdot}{G_x} \\ -k(\omega_x - \omega_x') + b(\omega_y - \omega_y') - i(\omega_z - \omega_z') &= \underset{\cdot}{G_y} \\ -j(\omega_x - \omega_x') - i(\omega_y - \omega_y') + c(\omega_z - \omega_z') &= \underset{\cdot}{G_z} \end{aligned}\right\} . \quad (6)$$

263. **Vis Viva of a Body having a Fixed Point.**—As the body has a fixed point, it is at any instant rotating round some axis through it; whence the vis viva is $I\omega^2$, I being the moment of inertia round the instantaneous axis.

Again, since $\frac{\omega_1}{\omega}, \frac{\omega_2}{\omega}, \frac{\omega_3}{\omega}$ are the direction cosines of the axis of rotation referred to the principal axes through the fixed point,

$$I = A\left(\frac{\omega_1}{\omega}\right)^2 + B\left(\frac{\omega_2}{\omega}\right)^2 + C\left(\frac{\omega_3}{\omega}\right)^2; \quad (\textit{Int. Calc.}, \text{Art. } 215);$$

whence, if $2T$ or S be the *vis viva* of the body, we have

$$2T = S = A\omega_1^2 + B\omega_2^2 + C\omega_3^2. \qquad (7)$$

If ω_x, ω_y, ω_z be the velocities of rotation, and a, b, c, i, &c., the moments and products of inertia of the body at any instant round space axes through O, the general equation of the momental ellipsoid referred to these axes leads to the following expression—

$$2T = I\omega^2 = a\omega_x^2 + b\omega_y^2 + c\omega_z^2 - 2i\omega_y\omega_z - 2j\omega_z\omega_x - 2k\omega_x\omega_y. \quad (8)$$

264. **Couple of Principal Moments.**—If a body be moving round a fixed point, we may imagine its actual velocity at any instant to be produced by an impulsive couple acting on it at the instant. By the last Article the components of this couple round the principal axes of the body are $A\omega_1$, $B\omega_2$, $C\omega_3$, and the axis of the couple is called the *Axis of Principal Moments.* This axis coincides at each instant with the momentum axis of the body (Arts. 210, 261).

If a tangent plane be drawn at the point of intersection of the instantaneous axis of rotation with the momental ellipsoid corresponding to the fixed point round which the body is rotating, the perpendicular from the centre on this tangent plane is the Axis of Principal Moments. This is obvious, when we remember that the direction cosines of this axis are proportional to $A\omega_1$, $B\omega_2$, $C\omega_3$; and those of the instantaneous axis of rotation to ω_1, ω_2, ω_3; and that the equation of the momental ellipsoid is

$$Ax^2 + By^2 + Cz^2 = K.$$

If ϕ be the angle between the momentum axis and the instantaneous axis of rotation, H the moment of momentum,

and S the *vis viva* of the body, we have, by the formula for the cosine of the angle between two lines in terms of their direction cosines,

$$H\omega\cos\phi = A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = S;$$

whence
$$\omega\cos\phi = \frac{S}{H}. \tag{9}$$

Again, if r be the intercept made by the momental ellipsoid on the instantaneous axis of rotation, we have

$$\frac{K}{r^2} = A\frac{\omega_1^2}{\omega^2} + B\frac{\omega_2^2}{\omega^2} + C\frac{\omega_3^2}{\omega^2} = \frac{S}{\omega^2};$$

whence
$$r^2 = \frac{K}{S}\omega^2. \tag{10}$$

Again, if we draw a tangent plane to the momental ellipsoid at the point where it meets the instantaneous axis of rotation, the intercept p made by this plane on the momentum axis is given by the equation $p = r\cos\phi$, since the momentum axis is perpendicular to the tangent plane. Hence, if we substitute for r and $\omega\cos\phi$ their values given by (10) and (9), we obtain

$$p = \frac{\sqrt{KS}}{H}. \tag{11}$$

Examples.

1. A body is set in motion by an impulsive couple whose magnitude is given; find the direction of its axis so that the initial *vis viva* of the body may be a maximum.

The axis of the couple must be the axis of least inertia of the body.

2. A body having a fixed point O is set in motion by an impulse, passing through a point P, which causes P to move with a velocity having a given magnitude and direction; determine the axis of instantaneous rotation.

Let the axis of x be a line through O in the direction of the velocity of P, and the axis of z the line OP; then, if V be the given velocity of P, h the distance OP, and ω_x, ω_y, ω_z the angular velocities of the body round the axes, we have $\omega_x = 0$, $h\omega_y = V$.

Now, by Thomson's Theorem, Art. 199, the value of ω_z must be such as to make T a minimum. But, Art. 263, (8),

$$\frac{dT}{d\omega_z} = -i\omega_y + c\omega_z. \quad \text{Hence} \quad c\omega_z = i\omega_y,$$

which determines ω_z, and consequently the axis of rotation.

3. In Ex. 2, when is the velocity of rotation around OP zero?

Ans. When OP is an axis of the section of the momental ellipsoid which is perpendicular to the initial motion of P.

4. In Ex. 2, if the magnitude of V be given as before, find its direction so that the initial *vis viva* of the body may be a maximum or a minimum.

By Art. 263, $$2T = \omega_x \frac{dT}{d\omega_x} + \omega_y \frac{dT}{d\omega_y} + \omega_z \frac{dT}{d\omega_z};$$

but

$$\omega_x = 0, \text{ and } \frac{dT}{d\omega_z} = 0 \text{ (Ex. 2).}$$

Hence

$$2T = b\omega_y^2 - i\,\omega_y\omega_z = \frac{bc - i^2}{c}\,\omega_y^2 = \frac{B'C'}{c}\frac{V^2}{h^2},$$

where B' and C' are the moments of inertia of the body round the axes of the section of the ellipsoid of inertia made by the plane yz. The maximum or minimum value of T is obtained, then, by making c equal to C' or to B'; *i. e.* the direction of V must be perpendicular to the central section of the ellipsoid having OP as an axis.

5. If a body be moving in any manner, the momentum axis, and the instantaneous axis of rotation through a given point O of the body, are the radius vector and the perpendicular on the corresponding tangent plane of the ellipsoid of gyration (see *Integral Calculus*, Art. 216) relative to O.

This is the reciprocal of the theorem given in Art. 264. It can be easily proved directly: σ_1, σ_2, σ_3, and α, β, γ, being the angles made by the momentum axis and the instantaneous axis of rotation with the principal axes, and a, b, c the principal radii of gyration,

$$H \cos\sigma_1 = A\omega_1 = ma^2\omega\cos\alpha, \text{ \&c.};$$

but if $x'y'z'$ be a point on the ellipsoid of gyration, α', β', γ' the angles made with the axes by the perpendicular on the corresponding tangent plane, and p the length of the perpendicular, $x'p = a^2\cos\alpha'$, &c.; therefore if x', &c. be proportional to $\cos\sigma_1$, &c., $\alpha' = \alpha$, &c.

The student will observe that H here denotes the moment of the momentum of the motion relative to O, but not of the absolute motion, except O be a point fixed in space.

6. If a tangent plane to the ellipsoid of gyration relative to any point of a body be drawn at right angles to the instantaneous axis of rotation passing throught the point, and ϕ be the angle between the instantaneous axis and the radius vector to the point of contact, $\omega = \dfrac{H\cos\phi}{mp^2}$.

This is immediately deducible from the consideration that $mp^2\omega = I\omega$ = moment of momentum round instantaneous axis of rotation $= H\cos\phi$.

7. Express the perpendicular on the tangent plane to the ellipsoid of gyration in terms of the angular velocity and relative *vis viva*.

Ans. $p^2 = \dfrac{S}{m\omega^2}$.

8. Express the intercept cut off by the ellipsoid of gyration on the momentum axis in terms of the relative *vis viva* and moment of momentum.

Ans. $R^2 = \dfrac{H^2}{mS}$.

9. A body, having a fixed point O, is set in motion by an impulse of given magnitude and passing through a given point P of the body; find the direction of the impulse so that the initial *vis viva* of the body may be a maximum.

Since the impulse has no moment round the line OP, the momentum axis lies in the plane perpendicular to OP: also by Ex. 8, the *vis viva* is a maximum when H is a maximum and R a minimum. Hence R must be the shortest axis of the section of the ellipsoid of gyration made by the plane perpendicular to OP, and the direction of the impulse must be parallel to the longest axis of this section.

10. In Ex. 9, show that the initial velocity of P in the direction of the impulse is a maximum, and that the instantaneous axis of rotation lies in the plane of R and OP.

Since the *vis viva* is a maximum, so is the initial velocity of P in the direction of the impulse, Art. 199. Again, if R' be the axis major of the section of the ellipsoid of gyration perpendicular to OP, the plane perpendicular to R' contains the perpendicular on the tangent plane drawn at the extremity of R, that is, the plane of R and OP contains the instantaneous axis of rotation.

265. **Motion of a Free Body under the Action of Impulses.**—If $\underset{.}{X}$, $\underset{.}{Y}$, $\underset{.}{Z}$, be the components of any one of the impulses, and $\bar{u}'$, $\bar{v}'$, $\bar{w}'$, $\bar{u}$, $\bar{v}$, $\bar{w}$ the components of the velocity of the centre of inertia G before and after the action of the impulses, the velocity of G is determined by the equations

$$\mathfrak{M}(\bar{u} - \bar{u}') = \Sigma \underset{.}{X}, \quad \mathfrak{M}(\bar{v} - \bar{v}') = \Sigma \underset{.}{Y}, \quad \mathfrak{M}(\bar{w} - \bar{w}') = \Sigma \underset{.}{Z}, \quad (12)$$

where $\mathfrak{M}$ is the mass of the body.

Again (Art. 209), the motion of the body relative to its centre of inertia is the same as if that point were fixed in space. Hence if $\underset{.}{L}$, $\underset{.}{M}$, $\underset{.}{N}$ be the moments of the impulses round the principal axes of the body at G, and ω_1', ω_2', ω_3', ω_1, ω_2, ω_3 the angular velocities of the body round these axes, before and after the action of the impulses, we have

$$A(\omega_1 - \omega_1') = \underset{.}{L}, \quad B(\omega_2 - \omega_2') = \underset{.}{M}, \quad C(\omega_3 - \omega_3') = \underset{.}{N}, \quad (13)$$

where A, B, C are the principal moments of inertia. Without having recourse to Art. 209, we may deduce equations (13) directly from (18), Art. 204, and (20), Art. 205, by the method of Ex. 2, Art. 213.

From equations (12) and (13) it appears that an impulse whose direction passes through the centre of inertia of a free rigid body produces a motion of translation only, whereas an impulse not passing through the centre of inertia produces both a translation and a rotation.

266. **General Expression for the Vis Viva of a Body.**—As the motion of a body relative to one of its points must always consist of a rotation round some axis through the point, it follows, from Art. 134, that if a body be free,

$$\Sigma mv^2 = \mathfrak{M} V^2 + I\omega^2,$$

where $\mathfrak{M}$ is the mass of the body; V the velocity of its centre of inertia; I the moment of inertia, and ω the angular velocity, round the instantaneous axis through the centre of inertia.

As was shown in Art. 263,

$$I\omega^2 = A\omega_1^2 + B\omega_2^2 + C\omega_3^2.$$

Again, if a, b, c, i, j, k be the moments and products of inertia for the centre of inertia, round three rectangular axes, which are parallel to fixed directions in space, and ω_x, ω_y, ω_z the corresponding angular velocities of the body,

$$I\omega^2 = a\omega_x^2 + b\omega_y^2 + c\omega_z^2 - 2i\omega_y\omega_z - 2j\omega_z\omega_x - 2k\omega_x\omega_y;$$

whence we have

$$\Sigma mv^2 = \mathfrak{M} V^2 + A\omega_1^2 + B\omega_2^2 + C\omega_3^2 \qquad (14)$$

$$= \mathfrak{M} V^2 + a\omega_x^2 + b\omega_y^2 + c\omega_z - 2i\omega_y\omega_z - 2j\omega_z\omega_x - 2k\omega_x\omega_y. \qquad (15)$$

June 7, 1873

EXAMPLES.

1. A free body is set in motion by an impulse. If the initial motion be a pure rotation, show that the directions of the impulse and of the instantaneous axis of rotation are principal axes of a section of the momental ellipsoid relative to the centre of inertia.

Since the initial motion is a pure rotation, the initial velocity of the centre of inertia is at right angles to the direction of the instantaneous axis of rotation. The above statement follows, then, from Art. 264.

2. On the same hypothesis as in the last example, show that the instantaneous axis of rotation is a principal axis of the body, at the point in which it is met by its shortest distance from the line of direction of the impulse (*see* Ex. 1, Art. 241).

3. If different impulses applied to the same body produce velocities of rotation round parallel instantaneous axes, prove that in general these axes lie in one plane containing the centre of inertia, and perpendicular to the lines of direction of the impulses, and that the points in which this plane meets these lines lie on a straight line.

4. If in the preceding example the instantaneous axes are parallel to a principal axis through the centre of inertia, prove that the lines of direction of the impulses lie in the corresponding principal plane at the centre of inertia.

The theory of the centre of percussion, given in Art. 235, may be collected from Examples 2, 3, 4.

5. A body is moving freely: under what circumstances can it be brought to rest by an impulse, and what must be the magnitude and position of the impulse?

The direction of the impulse must be opposite to that of the velocity of the centre of inertia, and its magnitude must be equal to the momentum of translation of the body. Again, the moment of the impulse round the centre of inertia must be equal and opposite to the couple of principal moments. Hence the magnitude and position of the impulse are determined, and the motion of the body must be such that the momentum axis is perpendicular to the direction of motion of the centre of inertia.

6. A free body is set in motion by an impulse of given magnitude, and passing through a given point P of the body; find the directions of the impulse for which the initial *vis viva* of the body is a minimum, and for which it is a maximum.

Since the impulse is given so is the velocity V of the centre of inertia; but the total *vis viva* $2T = m\,V^2 + S$, where S is the *vis viva* of the motion relative to the centre of inertia; hence T is a minimum when S is zero, *i. e.* when the direction of the impulse passes through the centre of inertia.

Again, the direction of the impulse for which S is a maximum is found as in Ex. 9, Art. 264, and when S is a maximum so likewise is T.

7. A free body is set in motion by an impulse whose magnitude and perpendicular distance from the centre of inertia of the body are given; find the direction of the impulse so that the initial *vis viva* of the body may be a maximum.

Here S must be a maximum, and therefore, as in Ex. 1, Art. 264, the impulse must lie in a plane passing through the centre of inertia and perpendicular to the axis of least inertia of the body.

267. Equations of Motion of a Body having a Fixed Point.—In the case of continuous forces, if G_x, G_y, G_z be the moments of the applied forces round the space axes, the equations of motion are (25), Art. 210,

$$\frac{dH_x}{dt} = G_x, \quad \frac{dH_y}{dt} = G_y, \quad \frac{dH_z}{dt} = G_z. \tag{16}$$

We may substitute for H_x, H_y, and H_z in these equations their values given by (1), or by (4). If we make the former substitution we obtain

$$\left.\begin{aligned} \frac{d}{dt}(a\omega_x - k\omega_y - j\omega_z) &= G_x \\ \frac{d}{dt}(-k\omega_x + b\omega_y - i\omega_z) &= G_y \\ \frac{d}{dt}(-j\omega_x - i\omega_y + c\omega_z) &= G_z \end{aligned}\right\} . \tag{17}$$

In the case of homogeneous spheres, as also in that of the initial motion of a body starting from rest, these equations are sometimes useful; but since in general a, k, j, &c. vary with the time, it is usually necessary to reduce equations (17) to a more manageable form.

If we substitute for H_x, H_y, H_z in (16) their values given by (4) and, after performing the differentiations, suppose the space axes to coincide with the instantaneous positions of the principal axes of the body at O, we have by (6), Art. 255, and (9), Art. 256, remembering that in this case a_2, a_3, b_1, b_3, c_1, c_2, are each zero, and that $a_1 = b_2 = c_3 = 1$,

$$\frac{dH_x}{dt} = A\frac{d\omega_1}{dt} - (B - C)\,\omega_2\,\omega_3,$$

$$\frac{dH_y}{dt} = B\frac{d\omega_2}{dt} - (C - A)\,\omega_3\,\omega_1,$$

$$\frac{dH_z}{dt} = C\frac{d\omega_3}{dt} - (A - B)\,\omega_1\,\omega_2\,;$$

whence, if L, M, N be the moments of the applied forces

round the principal axes at O, the equations of motion become

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3 &= L \\ B\frac{d\omega_2}{dt} - (C - A)\,\omega_3\omega_1 &= M \\ C\frac{d\omega_3}{dt} - (A - B)\,\omega_1\omega_2 &= N \end{aligned}\right\}. \tag{18}$$

Equations (18) are due to Euler, and are called by his name.

268. **Equations of Motion of a Free Body.**—If we denote the mass of the body by $\mathfrak{M}$, we have

$$\mathfrak{M}\frac{d^2\bar{x}}{dt^2} = \Sigma X, \quad \mathfrak{M}\frac{d^2\bar{y}}{dt^2} = \Sigma Y, \quad \mathfrak{M}\frac{d^2\bar{z}}{dt^2} = \Sigma Z, \tag{19}$$

where $\bar{x}$, $\bar{y}$, $\bar{z}$ are the coordinates of the centre of inertia referred to any three rectangular axes fixed in space; and ΣX, ΣY, ΣZ are the sums of the components of the applied forces parallel to these axes.

Again, since the motion of the body relative to its centre of inertia is the same as if that point were fixed in space (Art. 209), we have

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} - (B - C)\,\omega_2\omega_3 &= L \\ B\frac{d\omega_2}{dt} - (C - A)\,\omega_3\omega_1 &= M \\ C\frac{d\omega_3}{dt} - (A - B)\,\omega_1\omega_2 &= N \end{aligned}\right\}, \tag{20}$$

where ω_1, ω_2, ω_3 are the angular velocities; A, B, C the moments of inertia; and L, M, N the moments of the applied forces round the three principal axes of the body at the centre of inertia.

Instead of equations (20) we may use (17), the axes being parallels through the centre of inertia to directions fixed in space.

As in the case of impulses, equations (20) may be deduced directly from Art. 204 by the method of Ex. 2, Art. 213.

From equations (19) and (20), it appears that a force whose direction passes through the centre of inertia of a free body produces a motion of translation only, whereas a force not passing through the centre of inertia produces both a translation and a rotation.

Examples.

1. A body is given a rotation round a principal axis through its centre of inertia, and is acted on by a couple having this line for its axis. Show that the body will continue to revolve round the axis of initial rotation.

2. One end of a uniform rod rests on a horizontal plane and against a vertical wall; the other rests against a parallel vertical wall. All the surfaces being smooth, if the rod slips down, determine the motion.

Take the intersection of the horizontal and vertical planes passing through the first end of the rod for axis of x, and a vertical plane, at right angles to the walls and passing through the initial position of the centre of inertia of the rod, for the plane of yz, the axis of z being vertical.

Let β be the angle which the rod at any time makes with the axis of y, $2a$ its length, $2b$ the distance between the walls, x_1, y_1, z_1; x_2, y_2, z_2; and $\bar{x}, \bar{y}, \bar{z}$ the coordinates of the two extremities, and of the centre of inertia of the rod. Then

$$y_1 = 0, \quad z_1 = 0, \quad y_2 = 2b, \quad \bar{y} = \tfrac{1}{2}(y_1 + y_2) = b:$$

also $\bar{y} = a\cos\beta$, whence $\cos\beta = \dfrac{b}{a}$; thus, as β is constant, the motion of the rod relative to its centre of inertia is a rotation round the axis of y, whose amplitude at any time may be denoted by ϕ. Again, as $m\dfrac{d^2\bar{x}}{dt^2} = 0$, and as the initial value of $\dfrac{d\bar{x}}{dt}$ is zero, it is zero throughout the motion; also, since $\bar{y}$ is constant, $\dfrac{d\bar{y}}{dt} = 0$; whence the equation of *vis viva* is

$$m(k^2\dot{\phi}^2 + \dot{\bar{z}}^2) = 2mg(\bar{z}_0 - \bar{z}).$$

Now
$$k^2 = \frac{a^2\sin^2\beta}{3}, \quad \bar{z} = a\sin\beta\cos\phi,$$

whence, as the initial value of ϕ is zero, we obtain

$$(1 + 3\sin^2\phi)\,\dot{\phi}^2 = \frac{6g}{\sqrt{a^2 - b^2}}(1 - \cos\phi).$$

Also, $x_2 = a\sin\beta\sin\phi$, which determines the position of the upper end of the rod when ϕ is known.

3. **A heavy body is supported in equilibrium by two strings: one is cut; find the initial tension of the other.**

The two strings and the centre of inertia G of the body lie at first in the same vertical plane; let this plane be that of yz, the axis of z being vertical, and its positive direction downwards, and let the origin be the point O to which the uncut string is attached. (See figure, p. 293.)

Let l be the length of the string OA, and h the distance AG, the direction cosines of OA being α, β, γ, those of AG, λ, μ, ν; then, if x, y, z be the coordinates of G, we have $y = l\beta + h\mu$, $z = l\gamma + h\nu$; and, if T be the tension of the string, and m the mass of the body, the equations of motion of G are

$$m\ddot{x} = -T\alpha, \quad m\ddot{y} = -T\beta, \quad m\ddot{z} = mg - T\gamma.$$

Differentiating the expressions for y and z twice, substituting, and remembering that the initial values of the differential coefficients, with respect to the time, of α, β, γ, λ, μ, ν are each zero, we get

$$l\frac{d^2\beta}{dt^2} + h\frac{d^2\mu}{dt^2} = -\frac{T\beta}{m},$$

$$l\frac{d^2\gamma}{dt^2} + h\frac{d^2\nu}{dt^2} = g - \frac{T\gamma}{m}.$$

Multiplying the first of these equations by β, the second by γ, and adding, we have initially

$$h\left\{\beta\left(\frac{d^2\mu}{dt^2}\right) + \gamma\left(\frac{d^2\nu}{dt^2}\right)\right\} = g\gamma - \frac{T}{m},$$

since $\alpha^2 + \beta^2 + \gamma^2 = 1$, and initially $\alpha = 0$,

and therefore $$\beta\left(\frac{d^2\beta}{dt^2}\right) + \gamma\left(\frac{d^2\gamma}{dt^2}\right) = 0.$$

Now by (6), Art. 255, since λ is zero initially, we have

$$\frac{d^2\mu}{dt^2} = -\nu\frac{d\omega_x}{dt}, \quad \frac{d^2\nu}{dt^2} = \mu\frac{d\omega_x}{dt}.$$

Hence initially $$h(\beta\nu - \gamma\mu)\frac{d\omega_x}{dt} = \frac{T}{m} - g\gamma.$$

If p be the length of the perpendicular from G on the initial position of the string, this equation may be written

$$p\dot{\omega}_x = \frac{T}{m} - g\gamma.$$

Again, if a, b, c, i, j, k be the moments and products of inertia round axes through G parallel to the coordinate axes, we have initially

$$\frac{d}{dt}(a\omega_x - k\omega_y - j\omega_z) = -Tp,$$

$$\frac{d}{dt}(-k\omega_x + b\omega_y - i\omega_z) = 0,$$

$$\frac{d}{dt}(-j\omega_x - i\omega_y + c\omega_z) = 0.$$

In differentiating, since the initial values of ω_x, ω_y, and ω_z are each zero, a, &c., may be treated as constants, and as having the values belonging to the initial position of the body. We have, then, for the initial values of $\dot{\omega}_x$, &c., and T the equations,

$$a\dot{\omega}_x - k\dot{\omega}_y - j\dot{\omega}_z = -Tp,$$

$$-k\dot{\omega}_x + b\dot{\omega}_y - i\dot{\omega}_z = 0,$$

$$-j\dot{\omega}_x - i\dot{\omega}_y + c\dot{\omega}_z = 0;$$

whence, if Δ be the determinant

$$\begin{vmatrix} a & -k & -j \\ -k & b & -i \\ -j & -i & c \end{vmatrix},$$

we obtain $$\Delta\dot{\omega}_x = -(bc - i^2)\,Tp.$$

Substituting for $\dot{\omega}_x$, we have finally for T_0 the initial value o T,

$$T_0 = \frac{\Delta\gamma_0}{\Delta + mp^2(bc - i^2)}\,mg = \frac{ABC\gamma_0}{ABC + mp^2(bc - i^2)}\,mg.$$

269. **Motion of a Body round a Fixed Point, under the Action of no External Force.**—In this case equations (18) become

$$A\frac{d\omega_1}{dt} = (B - C)\,\omega_2\omega_3,\ B\frac{d\omega_2}{dt} = (C - A)\,\omega_3\omega_1,\ C\frac{d\omega_3}{dt} = (A - B)\,\omega_1\omega_2,$$

in which we shall suppose $A > B > C$.

Multiply the first by ω_1, the second by ω_2, the third by ω_3, add, and integrate, and we have

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = S. \qquad (21)$$

Next multiply the first by $A\omega_1$, the second by $B\omega_2$, the third by $C\omega_3$, add, and integrate, and we have

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = H^2. \tag{22}$$

In equations (21) and (22) S and H are constants. These equations could have been obtained directly from (3), Art. 261, and (7), Art. 263, by articles 200 and 213.

Again, if we multiply the first of the equations obtained from (18) by $\frac{\omega_1}{A}$, the second by $\frac{\omega_2}{B}$, the third by $\frac{\omega_3}{C}$, and add, we get

$$\omega_1\frac{d\omega_1}{dt} + \omega_2\frac{d\omega_2}{dt} + \omega_3\frac{d\omega_3}{dt} = \left(\frac{B-C}{A} + \frac{C-A}{B} + \frac{A-B}{C}\right)\omega_1\omega_2\omega_3,$$

or
$$\omega\frac{d\omega}{dt} = -\frac{(A-B)(B-C)(C-A)}{ABC}\omega_1\omega_2\omega_3.$$

If we combine the two equations already found with the equation

$$\omega_1^2 + \omega_2^2 + \omega_3^2 = \omega^2,$$

and solve for ω_1^2, we get

$$\begin{vmatrix} 1, & 1, & 1 \\ A, & B, & C \\ A^2, & B^2, & C^2 \end{vmatrix} \omega_1^2 = \omega^2 BC(C-B) - S(C^2-B^2) + H^2(C-B);$$

whence
$$\omega_1^2 = \frac{BC}{(A-B)(A-C)}\left\{\omega^2 - \frac{S(B+C)-H^2}{BC}\right\}. \tag{23}$$

If we denote $\frac{S(B+C)-H^2}{BC}$ by λ_1, and the two corresponding quantities by λ_2 and λ_3, we have

$$\omega_1\omega_2\omega_3 = \frac{ABC}{(A-B)(A-C)(B-C)}\sqrt{\{(\omega^2-\lambda_1)(\lambda_2-\omega^2)(\omega^2-\lambda_3)\}};$$

$$\therefore\quad \omega\frac{d\omega}{dt} = \sqrt{\{(\lambda_1-\omega^2)(\lambda_2-\omega^2)(\lambda_3-\omega^2)\}}. \tag{24}$$

Again, since

$$AS - H^2 = B(A - B)\omega_2^2 + C(A - C)\omega_3^2,$$

it follows that AS is always greater than H^2; in like manner we see that CS is less than H^2. Hence we see at once that λ_1, λ_2, and λ_3 are each positive quantities.

Also, we have

$$\lambda_2 - \lambda_1 = \frac{A-B}{ABC}(H^2 - CS), \quad \lambda_2 - \lambda_3 = \frac{B-C}{ABC}(AS - H^2);$$

therefore λ_2 is the greatest of the three; also $\lambda_1 - \lambda_3$ has the same sign as $BS - H^2$, and this depends on the initial conditions.

Again, since ω_1^2, ω_2^2, ω_3^2 are each positive,

$$\omega^2 > \lambda_1, \quad \omega^2 < \lambda_2, \quad \omega^2 > \lambda_3.$$

Hence we may assume either—

$$(1), \quad \omega^2 = \lambda_1 \sin^2\phi + \lambda_2 \cos^2\phi, \quad \text{or} \quad (2) \quad \omega^2 = \lambda_3 \sin^2\psi + \lambda_2 \cos^2\psi.$$

In the former case, if we select the negative sign of the square root in (24), that equation gives

$$\frac{d\phi}{dt} = \sqrt{\lambda_2 - \lambda_3 - (\lambda_2 - \lambda_1)\sin^2\phi} = \sqrt{\lambda_2 - \lambda_3}\sqrt{1 - k^2\sin^2\phi}, \quad (25)$$

where

$$k^2 = \frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_3}.$$

Hence, by (23), we get

$$\left.\begin{aligned} \omega_1^2 &= \frac{BC}{(A-B)(A-C)}(\lambda_2 - \lambda_1)\cos^2\phi \\ \omega_2^2 &= \frac{AC}{(A-B)(B-C)}(\lambda_2 - \lambda_1)\sin^2\phi \\ \omega_3^2 &= \frac{AB}{(A-C)(B-C)}(\lambda_2 - \lambda_3)(1 - k^2\sin^2\phi) \end{aligned}\right\}. \quad (26)$$

In the second case, we get

$$\frac{d\psi}{dt} = \sqrt{\lambda_2 - \lambda_1}\sqrt{1 - \frac{1}{k^2}\sin^2\psi}, \quad (27)$$

$$\left.\begin{aligned}\text{and}\qquad \omega_1^2 &= \frac{BC}{(A-B)(A-C)}(\lambda_2-\lambda_1)\left(1-\frac{1}{k^2}\sin^2\psi\right)\\ \omega_2^2 &= \frac{AC}{(A-B)(B-C)}(\lambda_2-\lambda_3)\sin^2\psi\\ \omega_3^2 &= \frac{AB}{(A-C)(B-C)}(\lambda_2-\lambda_3)\cos^2\psi\end{aligned}\right\}. \quad (28)$$

It is obvious that ϕ and ψ are connected by the equation $\sin\psi = k\sin\phi$.

We thus see that when either ϕ or ψ is known the values of ω_1, ω_2, ω_3 can be determined. Also, from (25) and (27), we see that ϕ and ψ are at once expressible in terms of t as elliptic functions of the first kind.

We now proceed to give a geometrical representation of the angles ϕ and ψ.

Let x, y, z be the coordinates of the point P in which the momentum axis, at any instant, intersects the surface of the ellipsoid of gyration,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1;$$

then, if $R = OP$, we have

$$A\omega_1 = H\frac{x}{R},\quad B\omega_2 = H\frac{y}{R},\quad C\omega_3 = H\frac{z}{R}.$$

Hence $$\frac{A\omega_1}{x}=\frac{B\omega_2}{y}=\frac{C\omega_3}{z}=\frac{H}{R}=\sqrt{\mathfrak{M}S};\ \text{(Ex. 8, Art. 264)}$$

$$\therefore\quad x=\frac{A}{\sqrt{\mathfrak{M}S}}\omega_1,\quad y=\frac{B}{\sqrt{\mathfrak{M}S}}\omega_2,\quad z=\frac{C}{\sqrt{\mathfrak{M}S}}\omega_3. \quad (29)$$

Hence, in terms of ϕ, we have

$$\left.\begin{aligned}x &= \sqrt{\frac{A}{\mathfrak{M}S}}\sqrt{\frac{ABC}{(A-B)(A-C)}}\sqrt{\lambda_2-\lambda_1}\cos\phi\\ y &= \sqrt{\frac{B}{\mathfrak{M}S}}\sqrt{\frac{ABC}{(A-B)(B-C)}}\sqrt{\lambda_2-\lambda_1}\sin\phi\end{aligned}\right\}. \quad (30)$$

And, in terms of ψ,

$$\left.\begin{aligned} y &= \sqrt{\frac{B}{\mathfrak{M}S}}\sqrt{\frac{ABC}{(A-B)(B-C)}}\sqrt{\lambda_2-\lambda_3}\sin\psi \\ z &= \sqrt{\frac{C}{\mathfrak{M}S}}\sqrt{\frac{ABC}{(A-C)(B-C)}}\sqrt{\lambda_2-\lambda_3}\cos\psi \end{aligned}\right\}. \qquad (31)$$

These equations show that the position of the momentum axis in the body is determined when either ϕ or ψ is known.

If we write equations (30) in the form

$$x = a\sqrt{\lambda_2-\lambda_1}\cos\phi, \quad y = \beta\sqrt{\lambda_2-\lambda_1}\sin\phi,$$

we get

$$\frac{x^2}{a^2} + \frac{y^2}{\beta^2} = \lambda_2 - \lambda_1. \qquad 32)$$

This is the equation to the projection on the plane of xy of the curve described by P on the surface of the ellipsoid of gyration. Next, if σ be the angle which a cyclic plane of the ellipsoid of gyration makes with the plane of xy, we easily see that

$$\cos\sigma = \frac{a}{b}\sqrt{\frac{b^2-c^2}{a^2-c^2}} = \sqrt{\frac{A}{B}\frac{(B-C)}{(A-C)}} = \frac{a}{\beta}.$$

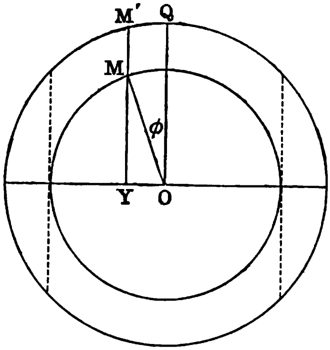
From this it follows that lines parallel to the axis of z will project the ellipse (32) into a circle on the cyclic plane. This, in fact, is a well-known theorem in surfaces of the second degree, since the locus of P is a sphero-conic. (Ex. 8, Art. 264.)

In like manner it follows from (31) that the projection of this sphero-conic on the cyclic plane by lines parallel to the axis of x is another circle.

These circles in the cyclic plane are exhibited in the accompanying figure, in which OY is the mean axis of the ellipsoid. If we suppose $\lambda_1 > \lambda_3$, *i.e.* $BS > H^2$, then it is easily seen that the inner circle is the projection by lines parallel to the axis of z. Hence, if M and M' be the positions of the projections of P at any instant, we shall have $MOQ = \phi$, and $M'OQ' = \psi$. This construction for the position of the momentum axis in the body is due to Mac Cullagh.

From Article 213 it appears that the direction of the momentum axis in space is invariable.

It should be observed that these results also hold good



for the motion of a free body under the action of no forces, provided its centre of inertia be taken as the origin.

Examples.

1. If $H^2 = BS$, find the position of the momentum axis at any instant.

It is immediately seen that in this case the momentum axis always lies in a cyclic plane of the ellipsoid of gyration. Also, since

$$\lambda_1 = \lambda_2, \quad \text{we have } k = 1.$$

Hence equation (25) becomes

$$\frac{d\phi}{\cos\phi} = \sqrt{\lambda_2 - \lambda_1}\, dt = h\,dt, \qquad \text{where } h = \sqrt{\frac{(A-B)(B-C)S}{ABC}}.$$

Hence

$$\log\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) = ht + \text{constant},$$

and we get

$$\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) = \tan\left(\frac{\pi}{4} + \frac{\phi_0}{2}\right) e^{ht},$$

where ϕ_0 is the initial value of ϕ. Hence the momentum axis, and therefore also the instantaneous axis of rotation, tends to approach without limit to the mean radius of gyration.

2. Investigate the motion if the initial axis of rotation be very close to the least axis of inertia.

In this case R is nearly equal to c, and $\therefore$ $H^2 - CS$ is very small: accordingly k is a very small quantity, and we get approximately, from equation (25),

$$\phi = \sqrt{\lambda_2 - \lambda_3}\, t + \phi_0 = \frac{H}{C}\sqrt{\frac{(B-C)(A-C)}{AB}}\, t + \phi_0,$$

where ϕ_0 is the initial value of ϕ.

If t_1 denote the time which the body takes to revolve round its axis of rotation (which is nearly coincident with the least axis of the ellipsoid of gyration), and t_2 the time of a complete revolution or oscillation of the momentum axis round the axis of z; then

$$\frac{H}{C}t_1 = 2\pi, \quad \frac{H}{C}\sqrt{\frac{(B-C)(A-C)}{AB}}\, t_2 = 2\pi, \quad R = c, \text{ approximately.}$$

Hence, approximately,

$$t_2 = t_1\sqrt{\frac{AB}{(A-C)(B-C)}}.$$

If $B - C$ be very small, t_2 will be very large in comparison with t_1.

A corresponding result may be obtained when R nearly $= a$.

This investigation would be applicable to the Earth if its axis of rotation were nearly but not exactly a principal axis. In this case t_1 would be the length of the day. The total attractions of other bodies are supposed to pass through the centre of inertia of the Earth.

3. If two of the principal moments of inertia of the body be equal, prove that —(1) the simultaneous positions of the momentum axis and the instantaneous axis of rotation lie in a plane containing the axis of unequal moment of inertia; (2) the instantaneous axis and the momentum axis describe in the body right circular cones whose semi-angles are i and γ, where

$$\tan^2 i = \frac{C}{A}\cdot\frac{H^2 - CS}{AS - H^2}, \quad \text{and} \quad \tan^2\gamma = \frac{A}{C}\cdot\frac{H^2 - CS}{AS - H^2},$$

the axis of unequal moment of inertia being the axis of z; (3) the values of $\omega, \omega_1, \omega_2, \omega_3$ at any time are given by the equations

$$\omega_1 = \omega \sin i \cos\left(\frac{A-C}{A}\omega_3 t + \chi\right), \quad \omega_2 = -\omega\sin i \sin\left(\frac{A-C}{A}\omega_3 t + \chi\right),$$

$$\omega_3 = \omega\cos i, \quad \omega^2 = \frac{(A+C)S - H^2}{AC}, \quad \text{where } \chi \text{ is an arbitrary constant.}$$

270. Conjugate Ellipsoid and Conjugate Line.— When a body on which no external force is acting is in motion round a fixed point, the squares of the angular

velocities of the body round its principal axes at the point must fulfil the two independent linear equations

$$\left.\begin{aligned} A\omega_1^2 + B\omega_2^2 + C\omega_3^2 - S = \Theta = 0 \\ A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 - H^2 = \Phi = 0 \end{aligned}\right\}. \qquad (33)$$

Any other linear equation, $\Theta' = 0$, between these variables must be of the form $\alpha\Theta + \beta\Phi = 0$, where α and β are constants, since otherwise each angular velocity would be completely determined. Hence, if we suppose that ω_1, ω_2, ω_3 satisfy also the two equations

$$\left.\begin{aligned} A'\omega_1^2 + B'\omega_2^2 + C'\omega_3^2 - S' = \Theta' = 0 \\ A'^2\omega_1^2 + B'^2\omega_2^2 + C'^2\omega_3^2 - H'^2 = \Phi' = 0 \end{aligned}\right\}, \qquad (34)$$

we must have $\Theta' = i(\lambda\Theta - \Phi)$, $\Phi' = j(\mu\Theta - \Phi)$, where i, λ, j, μ are constants.

Hence we get

$$\left.\begin{aligned} A' &= iA(\lambda - A), & A'^2 &= jA(\mu - A) \\ B' &= iB(\lambda - B), & B'^2 &= jB(\mu - B) \\ C' &= iC(\lambda - C), & C'^2 &= jC(\mu - C) \end{aligned}\right\}. \qquad (35)$$

From (35) we obtain

$$\frac{A(\lambda - A)^2}{\mu - A} = \frac{B(\lambda - B)^2}{\mu - B} = \frac{C(\lambda - C)^2}{\mu - C} = \frac{j}{i^2}; \qquad (36)$$

whence, by a well-known property of equal fractions, we have

$$\frac{A(\lambda - A)^2 - B(\lambda - B)^2}{A - B} = \frac{B(\lambda - B)^2 - C(\lambda - C)^2}{B - C} = -\frac{j}{i^2}; \qquad (37)$$

from these, by performing the divisions, we get

$$\lambda = \tfrac{1}{2}(A + B + C). \qquad (38)$$

Consequently, from (37) we obtain

$$2(AB+BC+CA)-A^2-B^2-C^2=\frac{4j}{i^2}, \tag{39}$$

and from (36) we get

$$\{2(AB+BC+CA)-A^2-B^2-C^2\}\mu=4ABC. \tag{40}$$

Finally, we have

$$\left.\begin{array}{l} A'=\frac{i}{2}A(B+C-A),\ B'=\frac{i}{2}B(C+A-B),\ C'=\frac{i}{2}C(A+B-C) \\ S'=i(\lambda S-H^2), \qquad H'^2=i^2\frac{ABC}{\mu}(\mu S-H^2) \end{array}\right\}, \tag{41}$$

where λ and μ have the values given by (38) and (40).

It appears from what has been said that any three constants a, b, c satisfying two equations of the form

$$a\omega_1^2+b\omega_2^2+c\omega_3^2=\text{constant},\ a^2\omega_1^2+b^2\omega_2^2+c^2\omega_3^2=\text{constant},$$

must be proportional either to A, B, C, or to A', B', C'. Hence, if we apply to A', B', C' a transformation similar to that which has been applied to A, B, C we must obtain quantities proportional to A, B, C. From this it follows that the two quadrics E and E' given by the equations

$$Ax^2+By^2+Cz^2=K,\ A'x^2+B'y^2+C'z^2=K',$$

are each derived from the other by a similar process, and may therefore be called *conjugate*. Since A, B, C are each positive, and such that the sum of any two is greater than the third, it follows from (38), (40), and (41), that A', B', C', λ, μ, S', and H'^2 are each positive. Hence we infer that the quadric E' is an ellipsoid.

If r' be the intercept made by the conjugate ellipsoid E' on the instantaneous axis of rotation of the body, p' the perpendicular from the fixed point on the tangent plane to E' at the extremity of r', and ϕ' the angle between r' and p', it can be proved in the same manner as in Art. 264, that

$$\omega\cos\phi'=\frac{S'}{H'},\ r'=\sqrt{\frac{K'}{S'}}\,\omega,\ p'=\frac{\sqrt{K'S'}}{H'}.$$

The perpendicular to the tangent plane to E' at the extremity

of r' corresponds to the momentum axis in the momental ellipsoid, and is called *the conjugate line.*

This Article and the following Examples are taken from a Paper by Dr. Routh in the *Quarterly Journal of Pure and Applied Mathematics* for 1888.

Examples.

1. If a body on which no external force is acting be moving round a fixed point O, and a quadric, having as axes the principal axes of the body at O, be such that the intercept which it makes on the instantaneous axis of rotation at any time is proportional to the angular velocity, and that the perpendicular from O on the tangent plane at the extremity of this intercept is constant, the quadric must be either the momental or the conjugate ellipsoid.

2. If P be a point on the conjugate line at a constant distance R from the fixed point O, and Q the point of the body which coincides at the instant with P, prove that the velocity of P is double that of Q, and that the directions of these two velocities coincide.

Let x, y, z be the coordinates of P referred to the principal axes at O; u, v, w its space velocities parallel to these axes; and u', v', w' those of Q; then

$$u' = \omega_2 z - \omega_3 y, \quad v' = \omega_3 x - \omega_1 z, \quad w' = \omega_1 y - \omega_2 x,$$

$$u = \dot{x} + u', \qquad v = \dot{y} + v', \qquad w = \dot{z} + w'.$$

Now,

$$H'x = RA'\omega_1, \quad H'y = RB'\omega_2, \quad H'z = RC'\omega_3;$$

hence, by (41), we have

$$\dot{x} = \frac{iR}{2H'} A(B + C - A)\dot{\omega}_1,$$

and

$$u' = \frac{iR}{2H'}(B - C)(B + C - A)\omega_2\omega_3;$$

whence, by Euler's equations, Art. 267, we obtain $\dot{x} = u'$, and therefore $u = 2u'$; and in like manner $v = 2v'$, $w = 2w'$.

3. Determine the motion of the conjugate line in space.

Let θ be the angle between the conjugate line OP and the invariable line or momentum axis OZ, ψ the angle which the plane ZOP makes with a fixed plane passing through OZ, ϕ and ϕ' the angles made with OZ and OP by the instantaneous axis of rotation of the body; also let Ω be the component round OZ of the angular velocity of the body, and Ω' its component round OP; then

$$\Omega = \omega\cos\phi = \frac{S}{H}, \quad \Omega' = \omega\cos\phi' = \frac{S'}{H'}.$$

By considering the motion of a point of the body situated at the instant on OP, it is plain from Ex. 2, that the angular velocity of the body round an axis perpendicular to OP in the plane ZOP is $\frac{1}{2}\sin\theta\dot{\psi}$. Hence

$$\Omega = \tfrac{1}{2}\sin^2\theta\dot{\psi} + \Omega'\cos\theta,$$

whence $$\sin^2\theta\dot\psi = 2\left(\frac{S}{H} - \frac{S'}{H'}\cos\theta\right). \qquad (a)$$

Again, the whole velocity of a point of the body at the unit distance from O on the line OP being $\omega\sin\phi'$, we have, by Ex. 2,

$$\tfrac{1}{4}(\sin^2\theta\dot\psi^2 + \dot\theta^2) = \omega^2\sin^2\phi',$$

that is

$$\sin^2\theta\dot\psi^2 + \dot\theta^2 = 4\left(\omega^2 - \frac{S'^2}{H'^2}\right). \qquad (b)$$

We have now to express ω^2 in terms of θ, which can be done as follows:— Expressing $\cos\theta$ in terms of the direction cosines of OP and OZ, we have

$$HH'\cos\theta = AA'\omega_1^2 + BB'\omega_2^2 + CC'\omega_3^2$$
$$= i\lambda\,(A^2\omega_1^2 + B_2^2\omega_2^2 + C^2\omega_3^2) - i\,(A^3\omega_1^2 + B^3\omega_2^2 + C^3\omega_3^2).$$

Hence we get

$$A^3\omega_1^2 + B^3\omega_2^2 + C^3\omega_3^2 = \lambda H - \frac{HH'}{i}\cos\theta.$$

If we combine this equation with (33), and solve for ω_1^2, we obtain

$$(A-B)(B-C)(C-A)\,\omega_1^2 = -\frac{B-C}{A}\left\{SBC + H^2(A-\lambda) - \frac{HH'}{i}\cos\theta\right\}.$$

From this and the similar expressions for ω_2^2 and ω_3^2 we get

$$ABC\,\omega^2 = S(AB + BC + CA) - \tfrac{1}{2}H^2(A+B+C) - \frac{HH'}{i}\cos\theta.$$

Substituting the value for ω^2 given by this equation in (b), we obtain

$$\sin^2\theta\,\dot\psi^2 + \dot\theta^2 = \frac{4}{ABC}\{S(AB+BC+CA) - \tfrac{1}{2}H^2(A+B+C)\}$$
$$- 4\left(\frac{S'}{H'}\right)^2 - \frac{4}{i\,ABC}HH'\cos\theta.$$

From (a) and this equation θ and ψ can be obtained by quadratures.

271. Stress Exerted by a Body on a Fixed Point.—In order to determine the force exerted by a fixed point on a body we have only to consider the point as replaced by a force, whose components are X_0, Y_0, Z_0, passing through it. We may then consider the body as free, and we have, by Article 268,

$$\mathfrak{M}\frac{d^2\bar{x}}{dt^2} = \Sigma X + X_0,$$

with two similar equations.

But as the body is rotating round the origin, if we suppose the axes fixed in space to coincide at the instant under

consideration with the principal axes through the origin, we have

$$\frac{d^2\bar{x}}{dt^2} = -\bar{y}\frac{d\omega_3}{dt} + \bar{z}\frac{d\omega_2}{dt} + \omega_1(\bar{y}\omega_2 + \bar{z}\omega_3) - (\omega_2^2 + \omega_3^2)\bar{x}.$$

Substituting for $\frac{d\omega_3}{dt}$ and $\frac{d\omega_2}{dt}$ from Euler's Equations, we get,

$$\frac{d^2\bar{x}}{dt^2} = -\bar{y}\frac{N}{C} + \bar{z}\frac{M}{B} + \omega_1(B + C - A)\left(\bar{y}\frac{\omega_2}{C} + \bar{z}\frac{\omega_3}{B}\right) - (\omega_2^2 + \omega_3^2)\bar{x}.$$

Now, let S_1, S_2, S_3 be the components of the stress on the fixed point at any time, in the directions occupied at the instant by the principal axes of the body, then $S_1 = -X_0$, and therefore

$$\left.\begin{aligned}
S_1 &= \Sigma X - \mathfrak{M}\left\{-\bar{\eta}\frac{N}{C} + \bar{\zeta}\frac{M}{B} + \omega_1(B + C - A)\left(\bar{\eta}\frac{\omega_2}{C} + \bar{\zeta}\frac{\omega_3}{B}\right) - (\omega_2^2 + \omega_3^2)\bar{\xi}\right\}\\
S_2 &= \Sigma Y - \mathfrak{M}\left\{-\bar{\zeta}\frac{L}{A} + \bar{\xi}\frac{N}{C} + \omega_2(C + A - B)\left(\bar{\zeta}\frac{\omega_3}{A} + \bar{\xi}\frac{\omega_1}{C}\right) - (\omega_3^2 + \omega_1^2)\bar{\eta}\right\}\\
S_3 &= \Sigma Z - \mathfrak{M}\left\{-\bar{\xi}\frac{M}{B} + \bar{\eta}\frac{L}{A} + \omega_3(A + B - C)\left(\bar{\xi}\frac{\omega_1}{B} + \bar{\eta}\frac{\omega_2}{A}\right) - (\omega_1^2 + \omega_2^2)\bar{\zeta}\right\}
\end{aligned}\right\}, \quad (42)$$

where $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ are the coordinates of the centre of inertia referred to the principal axes through the fixed point, and are absolute constants: ΣX is the sum of the components of the applied forces parallel to one of these axes, and L the moment round it of the same forces. $\Sigma X, \Sigma Y, \Sigma Z, L, M, N$ are in general variable with the time.

In like manner if $\dot{S}_1$, $\dot{S}_2$, $\dot{S}_3$ be the impulses arising from the instantaneous stresses exerted by a body on a fixed point, in consequence of the action on the body of any system of impulses, we obtain, by Arts. 255 and 265,

$$\left.\begin{aligned}
\underset{\cdot}{S}_1 &= \Sigma \underset{\cdot}{X} + \mathfrak{M}\left\{\bar{\eta}\frac{\underset{\cdot}{N}}{C} - \bar{\zeta}\frac{\underset{\cdot}{M}}{B}\right\}\\
\underset{\cdot}{S}_2 &= \Sigma \underset{\cdot}{Y} + \mathfrak{M}\left\{\bar{\zeta}\frac{\underset{\cdot}{L}}{A} - \bar{\xi}\frac{\underset{\cdot}{N}}{C}\right\}\\
\underset{\cdot}{S}_3 &= \Sigma \underset{\cdot}{Z} + \mathfrak{M}\left\{\bar{\xi}\frac{\underset{\cdot}{M}}{B} - \bar{\eta}\frac{\underset{\cdot}{L}}{A}\right\}
\end{aligned}\right\}. \quad (43)$$

272. **Centrifugal Couple.**—If a body have a fixed point O, the change produced in its angular velocity round one of its principal axes at O in the element of time dt is given, (18), Art. 267, by the equation

$$A d\omega_1 = (B - C)\, \omega_2 \omega_3 \, dt + L\, dt.$$

The first term on the right-hand side of this equation results from the angular velocities already existing round the other two axes. In consequence of these velocities each point of the body, in virtue of its connexions with the other points, exerts a force on the entire body. These forces are in fact the centrifugal forces resulting from the motion of the body, and their moments L', M', N' round axes fixed in space may be determined directly as follows:—

Let α, β, γ be the angles which the instantaneous axis of rotation makes with the axes of coordinates; p the perpendicular distance from this axis to any point xyz of the body; q the intercept between the origin and the foot of p; r the radius vector to the point xyz; and ω the angular velocity of the body round the instantaneous axis. The centrifugal force at the point xyz is $mp\omega^2$ acting along p; and the component of this force along the axis of x is $m\omega^2$ multiplied by the projection of p.

If we project the triangle formed by rpq on the axis of x, we have

projection of p = projection of r − projection of $q = x - q \cos \alpha$,

and $$q = x \cos \alpha + y \cos \beta + z \cos \gamma;$$

hence the centrifugal force along axis of x

$$= m\omega^2 \{x - (x \cos \alpha + y \cos \beta + z \cos \gamma) \cos \alpha\}$$

$$= m\omega^2 \{x (\cos^2 \beta + \cos^2 \gamma) - y \cos \alpha \cos \beta - z \cos \alpha \cos \gamma\}$$

$$= m \{x (\omega_y^2 + \omega_z^2) - y \omega_x \omega_y - z \omega_x \omega_z\},$$

remembering that

$$\omega_x = \omega \cos \alpha, \quad \omega_y = \omega \cos \beta, \quad \omega_z = \omega \cos \gamma.$$

In like manner for the force along the axis of y, we have

$$m\{y(\omega_z^2 + \omega_x^2) - z\omega_y\omega_z - x\omega_y\omega_x\},$$

and for that along the axis of z,

$$m\{z(\omega_x^2 + \omega_y^2) - x\omega_z\omega_x - y\omega_z\omega_y\};$$

whence, taking moments round the axis of x, and integrating through the entire body, we obtain

$$L' = (\omega_y^2 - \omega_z^2)\int yzdm + \omega_y\omega_z\int(z^2 - y^2)\,dm - \omega_z\omega_x\int xydm$$
$$+ \omega_x\omega_y\int xz\,dm. \qquad (44)$$

If we now suppose the axes to coincide with the instantaneous positions of the principal axes of the body, every term in L' vanishes except $\omega_y\omega_z\int(z^2 - y^2)\,dm$, and we get

$$L' = (B - C)\,\omega_2\omega_3. \qquad (45)$$

Accordingly, the couple whose components round the three axes are $(B - C)\,\omega_2\omega_3$, &c., is called the *centrifugal couple.*

The axis of the centrifugal couple is at right angles to the axis of principal moments, and to the axis of rotation.

For the direction cosines of the axis of the centrifugal couple are proportional to

$$(B - C)\,\omega_2\omega_3, \quad (C - A)\,\omega_3\omega_1, \quad (A - B)\,\omega_1\omega_2;$$

whence it is seen at once that the conditions for its being perpendicular to the two other lines are fulfilled.

If a central section of the momental ellipsoid be taken passing through the instantaneous axis of rotation and the axis of the centrifugal couple, these two lines coincide with the principal axes of this section.

The lines in question are at right angles, and one is parallel to the tangent plane through the point where the other intersects the ellipsoid.

273. **Motion of a Free Body relative to its Centre of Inertia.**—As the equations for determining the motion of a body relative to its centre of inertia are the same as if the centre of inertia were a fixed point, the theorems of Arts. 264 and 272, in reference to the instantaneous axis of rotation, the axis of the centrifugal couple, and the axis of principal moments, hold good.

Examples.

Motion of a Body unacted on by Force.

1. The angular velocity at any instant is proportional to the intercept on the instantaneous axis of rotation through the centre of inertia cut off by the momental ellipsoid.

The velocity of the centre of inertia is constant as well as the whole *vis viva*. Hence the *vis viva* of the motion relative to the centre of inertia is constant, and therefore, (10), Art. 264, ω is proportional to r.

2. The component of the angular velocity round the momentum axis through the centre of inertia is constant. *See* (9), Art. 264.

3. If a tangent plane be drawn to the momental ellipsoid at its point of intersection with the instantaneous axis of rotation through the centre of inertia, the distance of this plane from the centre is constant.

This follows from (11), Art. 264.

If a body have a fixed point, the results of the preceding examples hold good, the fixed point being substituted for the centre of inertia.

4. A body moves round a fixed point: give a geometrical representation of the motion.

The momental ellipsoid relative to the point rolls on a plane fixed in space, so that the line joining the centre to the point of contact is always the instantaneous axis of rotation.

5. A body is moving round a fixed point; find the locus of the instantaneous axis of rotation in the body.

Since $\frac{\omega_1}{\omega}, \frac{\omega_2}{\omega}, \frac{\omega_3}{\omega}$ are its direction cosines referred to the principal axes through the point, its locus is the cone

$$A(H^2 - AS)x^2 + B(H^2 - BS)y^2 + C(H^2 - CS)z^2 = 0.$$

6. Find the locus of the momentum axis in the body.

Its locus is the cone

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = \frac{S}{H^2}(x^2 + y^2 + z^2), \quad \text{or}$$

$$\frac{AS - H^2}{A}x^2 + \frac{BS - H^2}{B}y^2 + \frac{CS - H^2}{C}z^2 = 0.$$

Hence the curve traced out by this line on the ellipsoid of gyration is a sphero-conic, as already stated in Art. 269.

7. Determine the curve traced out on the momental ellipsoid by the instantaneous axis.

The equations of the curve are got by combining the equations of the ellipsoid with that of the cone given in Ex. 5; they are, therefore,

$$Ax^2 + By^2 + Cz^2 = K, \quad A^2x^2 + B^2y^2 + C^2z^2 = \frac{KH^2}{S}.$$

This curve is called the *polhode.*

The curve traced out on the fixed plane, by the point of contact, is called the *herpolhode.*

8. The projections of the polhode on the planes perpendicular to the axes of greatest and least moment of inertia are ellipses. Its projection on the plane perpendicular to the remaining principal axis is a hyperbola.

This appears at once by eliminating x, y, z successively from the two equations of Ex. 7, remembering that $A > B > C$.

9. In what case does the hyperbola become a pair of straight lines?

If $H^2 = BS$. (*See* Ex. 1, Art. 269.)

10. If the body be free, give a geometrical representation of the motion. (*See* Ex. 4.)

The momental ellipsoid relative to the centre of inertia rolls on a plane at a constant distance from the centre of inertia and parallel to a plane fixed in space, the instantaneous axis of rotation being the line joining the centre of inertia to the point of contact, whilst the whole system moves with uniform velocity parallel to a fixed direction.

11. Show that the *herpolhode* lies between two circles the squares of whose radii are

$$\frac{K}{S}\left(\lambda_2 - \frac{S^2}{H^2}\right), \text{ and } \frac{K}{S}\left(\lambda_1 - \frac{S^2}{H^2}\right), \text{ or } \frac{K}{S}\left(\lambda_3 - \frac{S^2}{H^2}\right)$$

according as (*see* Art. 269) λ_1 is greater or less than λ_3.

If ρ be the distance from the point of contact to the foot of the perpendicular on the fixed plane, we have

$$\rho^2 = r^2 - p^2\text{; but } p^2 = \frac{KS}{H^2}, \text{ and } r^2 = \frac{K}{S}\omega^2 \text{ (Art. 264); } \therefore \rho^2 = \frac{K}{S}\left(\omega^2 - \frac{S^2}{H^2}\right).$$

But, Art. 269, $\omega^2 > \lambda_1$, $\omega^2 < \lambda_2$, $\omega^2 > \lambda_3$.

Hence the greatest and least values of ρ^2 are comprised between the limits stated above.

12. If a body be rotating round a fixed point, or a free body round its centre of inertia, the couple resulting from centrifugal forces lies in the plane containing the momentum axis and the instantaneous axis of rotation, and its magnitude is $H\omega \sin\phi$, or $S \tan\phi$, where ϕ is the angle between the instantaneous and the momentum axes.

The components of the couple resulting from centrifugal forces are (Art. 272)

$$(B - C)\,\omega_2\,\omega_3, \quad (C - A)\,\omega_3\,\omega_1, \quad (A - B)\,\omega_1\,\omega_2,$$

or $m\omega^2(b^2 - c^2)\cos\beta\cos\gamma$, $m\omega^2(c^2 - a^2)\cos\gamma\cos\alpha$, $m\omega^2(a^2 - b^2)\cos\alpha\cos\beta$;

where α, β, γ are the angles made by the instantaneous axis of rotation with the principal axes of the body, and a, b, c are the semi-axes of the ellipsoid of gyration. If p be the perpendicular from the origin on the tangent plane to the ellipsoid of gyration at the point $x'y'z'$ where it is met by the momentum axis R, double the projection of the triangle formed by the origin, $x'y'z'$, and the foot of p, is

$$p(x'\cos\beta - y'\cos\alpha), \quad \text{or} \quad (a^2 - b^2)\cos\alpha\cos\beta,$$

and double the area of the same triangle is $Rp\sin\phi$; therefore by Ex. 5, 7, 8, Art. 264, we have the required result.

13. If a tangent plane be drawn to the ellipsoid of gyration at the point where it is met by the axis of the centrifugal couple, the perpendicular on this tangent plane is the axis of the rotation produced by the centrifugal couple.

L', M', N' being the components of the centrifugal couple, and $\delta\omega_1$, $\delta\omega_2$, $\delta\omega_3$, the rotations produced by it *considered alone*, we have, from Euler's equations,

$$A\delta\omega_1 = L'dt, \quad B\delta\omega_2 = M'dt, \quad C\delta\omega_3 = N'dt;$$

but these equations are of the same form as those connecting the instantaneous axis with the components of the couple of principal moments; therefore, &c.

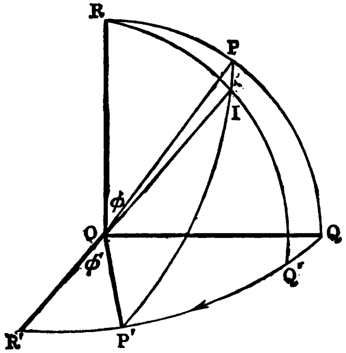
It follows from this, that the axis of rotation produced by the centrifugal couple is at right angles to the momentum axis; for (*see* Fig., Ex. 16) if OR be the momentum axis; OP the instantaneous axis of rotation; OR' the axis of the centrifugal couple, and OP' the axis of the centrifugal couple rotation; OR' being at right angles to OP (Ex. 12), is conjugate to OR: hence OR is parallel to the tangent plane through R', and therefore at right angles to OP'. Also, OR and OR' are the principal axes of the section of the ellipsoid made by their plane.

14. The intercept on the momentum axis cut off by the ellipsoid of gyration is of constant length (Ex. 8, Art. 264).

15. The motion of the momentum axis in the body consists of a series of rotations, the axis of each rotation being at right angles both to the momentum axis and the centrifugal couple axis, and the magnitude of the rotation being equal and opposite to the rotation of the body round the same axis.

The centrifugal couple tends at each instant to alter the position of the momentum axis, since the new moment of momentum is the resultant of the principal couple at the beginning of the instant and the momentum produced by the centrifugal couple during the instant. The former component is H, the latter $H\omega\sin\phi dt$ (Ex. 12), and the two are at right angles. Hence the momentum axis OR turns towards the centrifugal couple axis OR' with an angular velocity $\omega\sin\phi$, which is equal and opposite to the angular velocity of the body round OQ, the perpendicular to the momentum axis and the centrifugal couple axis.

16. The axis of the centrifugal couple, regarded as a radius vector of the ellipsoid of gyration, describes areas proportional to the time in the invariable plane of principal moments.



Let OR be the momentum axis and OR' the centrifugal couple axis at any instant. Let the lengths of OR and OR', regarded as radii vectores of the ellipsoid of gyration, be R and r. Describe a sphere with radius R round O as centre, and let OP and OP' be the positions of the actual axis of rotation and of the axis of rotation due to centrifugal forces at the instant. Let the body be rotating clockwise round OP. At the end of the time dt the instantaneous axis of rotation will have turned towards OP', into the position OI.

In the figure the line OR and the plane at right angles to it are fixed in space, whilst OP and OI are consecutive positions in space of the instantaneous axis of rotation. The position of the centrifugal couple axis at the end of the time dt is at right angles to the plane RIQ'. If dv be the angle described by this axis in the time dt, $dv = QOQ'$. If ω' be the angular velocity produced in the time dt by the centrifugal couple, we have (Ex. 12 and 13, and Ex. 6, Art. 264)

$$m\omega' p'r = m\omega^2 pR \sin\phi\, dt;$$

whence $$\frac{\omega R p \sin\phi\, dt}{rp'} = \frac{\omega'}{\omega} = \frac{\sin PI}{\sin P'I} = \frac{\sin PI}{\sin IR} \cdot \frac{\sin IR}{\sin P'I}$$

$$= \frac{dv}{\sin P'PQ} \cdot \frac{\sin RP}{\sin P'P} = \frac{dv \sin\phi}{\cos\phi'};$$

but $$p' = r\cos\phi', \quad \text{and} \quad p = R\cos\phi;$$

whence, finally, $$r^2\, dv = \omega R^2 \cos\phi\, dt;$$

and as R and $\omega\cos\phi$ are each constant, the theorem is proved. It is to be observed that it follows from the equations of Ex. 13 above, and Ex. 5, Art. 264, that the angles ROP and $R'OP'$ are each acute.

Another mode of proving the theorem contained in this example is as follows :—

As before, $$\frac{\omega'}{\omega} = \frac{dv \sin\phi}{\cos\phi'};$$

therefore $$\frac{\omega'}{dt} = \frac{dv}{dt}\,\frac{\omega\sin\phi}{\cos\phi'}.$$

Again, from Exs. 12 and 13, we readily see that

$$\frac{\omega'}{dt} = \frac{S\tan\phi\cos\phi'}{mp'^2} = \frac{S\tan\phi}{mr^2\cos\phi'},\ \text{since } p' = r\cos\phi'.$$

Hence, equating these values of $\frac{\omega'}{dt}$, we get

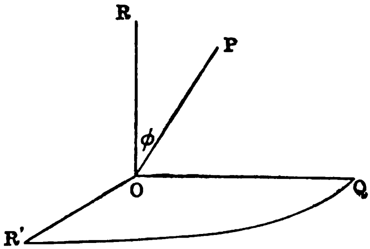
$$r^2\frac{dv}{dt} = \frac{S}{m\omega\cos\phi} = \frac{H}{m}\quad\text{by (9), Art. 264.}$$

17. To determine the position of the body in space at any time.

The line in the body which at a given time coincides with the momentum axis is known from Art. 269. If, then, we make this line of the body coincide with the momentum axis (whose position in space remains unaltered), and then turn the body through the proper angle round this axis, the position of the body in space is determined.

To effect the latter part of the determination, we consider the position in space of the line OQ, which is at right angles to the momentum axis and the axis of the centrifugal couple.

The momentum axis OR describes a cone C in the body, and it is easily seen that OQ describes the reciprocal cone C'. The position of the momentum axis in the body being known, so likewise is the position of OQ, the corresponding edge of C'; and if we can determine the position of the latter in space, the

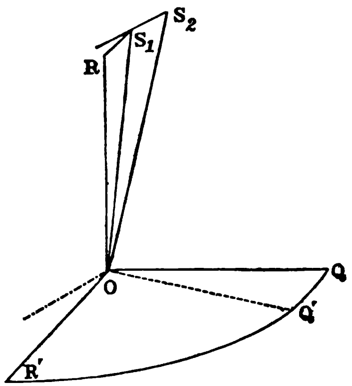


problem is solved. The instantaneous axis of rotation OP lies in the plane of OR and OQ; and the angular velocity ω round OP is equivalent to $\omega\cos\phi$ round OR, and $\omega\sin\phi$ round OQ. If, then, we suppose the cone C' rigidly connected with the body, the whole motion consists of the rolling and sliding of the cone

C' on the plane of principal moments. The angular velocity of sliding is $\omega \cos\phi$; and if $d\epsilon$ be the angle between two consecutive edges of the cone C', the velocity with which OQ turns in consequence of the rolling is $\frac{d\epsilon}{dt}$. Hence, if the cone C' be on the same side of the plane of principal moments as OP, the whole angular velocity of the edge round OR is

$$\omega \cos\phi - \frac{d\epsilon}{dt}.$$

Thus, if the rotation round OP be counter-clockwise, $\omega \cos\phi$ will be from R' to Q, whilst the rolling round OQ will bring an edge into contact with the plane of principal moments which is nearer to OR', and this will impart an angular velocity $-\frac{d\epsilon}{dt}$ to the edge of C' which is in the plane of principal moments. We



can arrive at the same result in another way, by considering the motion of the cone C regarded as rigidly attached to the body. The rotation round OQ (supposed counter-clockwise) brings the line OS_1 consecutive to OR into the position OR, and moves the next consecutive tangent plane of C in a direction parallel to OR'. The next rotation is effected round a line at right angles to $OS_1 S_2$, which is therefore nearer than OQ to OR': thus the motion of OQ, in consequence of a counter-clockwise rotation round OQ, is clockwise. On the other hand, the motion of OQ, in consequence of a counter-clockwise rotation round OR, is likewise counter-clockwise.

Hence, on the whole, if v be the angle described by the line at right angles to the momentum axis and the centrifugal couple axis in the fixed plane,

$$v = \int \omega \cos\phi \, dt - \int \frac{d\epsilon}{dt} dt.$$

Since $d\epsilon$ is the angle between two consecutive edges of the cone C',

$$d\epsilon = \frac{ds}{R},$$

if s be the arc of the spherical conic in which the cone C' meets the sphere of radius R; whence, finally,

$$v = \omega t \cos\phi - \frac{s}{R}.$$

If the cone C' be on the opposite side of the plane of principal moments from OP and OR, or, in other words, if the curvature of the cone C turned towards OP be convex instead of concave, the two parts of the motion of OQ have the same sign, and

$$v = \omega t \cos\phi + \frac{s}{R}.$$

The former case occurs when R is less than the mean axis of the ellipsoid of gyration; the latter, when it is greater. In either case s is determined by knowing the position of OQ in the cone C'.

In order to facilitate the drawing of the figures, the rotation is in Ex. 16 supposed to be clockwise, but in the present example counter-clockwise.

It is to be observed also that in the figure of the present Example the angle QOQ' represents only part of the motion of OQ, viz. $d\epsilon$, while in the figure of Ex. 16 it represents the whole motion dv.

Examples 12 to 16 are due to Mac Cullagh.

18. The normals to the cone described by the instantaneous axis of rotation intersect the ellipsoid of gyration in a line of curvature.

The normal to the plane of OP and OP' (figure of Ex. 16) is conjugate to OR and OR', and hence passes through the intersection of the ellipsoid of gyration with the confocal $a^2 - R^2$. This result follows also from Ex. 5.

19. Show that, if v and ϕ have the same meaning as in Ex. 16,

$$\frac{dv}{dt} = \frac{S}{H} + \frac{(AS - H^2)(BS - H^2)(CS - H^2)}{ABCHS^2} \cot^2\phi.$$

Substituting, by (9) and Ex. 8, Art. 264,

$$\omega \cos\phi \text{ for } \frac{S}{H},\ R^2 \text{ for } \frac{H^2}{mS}, \text{ and } ma^2 \text{ for } A,\ mb^2 \text{ for } B,\ mc^2 \text{ for } C,$$

the equation given above is reduced to

$$\frac{dv}{dt} = \omega \cos\phi \left\{1 + \frac{(a^2 - R^2)(b^2 - R^2)(c^2 - R^2)}{a^2 b^2 c^2} \cot^2\phi\right\}.$$

Now, by Ex. 16, $$\frac{dv}{dt} = \omega \cos\phi \frac{R^2}{r^2};$$

also $$\frac{R^2}{r^2} = 1 + \frac{(a^2 - R^2)(b^2 - R^2)(c^2 - R^2)}{a^2 b^2 c^2} \cot^2\phi,$$

as may be thus proved.

The axes of a central section of the quadric a are parallel to the normals to the two confocals through the extremities of the semi-diameter D conjugate to the section, and (a', a'' being the semi-axes of the confocals) are given by the equations

$$R^2 = a^2 - a'^2, \quad r^2 = a^2 - a''^2$$

(Salmon's *Geometry of Three Dimensions*, § 164). Moreover, if r' be the axis normal to a'' of the section conjugate to D in the quadric a', the direction of r' coincides with that of r, and the magnitude of r' is given by the equation

$$r'^2 = a'^2 - a''^2 = r^2 - R^2.$$

Again, if p' be the perpendicular on the tangent plane to a', which is parallel to the plane of D and r' or r,

$$p'^2 = a'^2 \cos^2 \alpha + b'^2 \cos^2\beta + c'^2 \cos^2 \gamma = p^2 - R^2,$$

since $\qquad a'^2 = a^2 - R^2, \quad b'^2 = b^2 - R^2, \quad c'^2 = c^2 - R^2.$

Hence, if ψ be the angle between D and the direction of r or r', we have, from the quadric a,

$$D^2 = \frac{a^2 b^2 c^2}{r^2 p^2 \sin^2\psi}$$

and, from the quadric a',

$$D^2 = \frac{a'^2 b'^2 c'^2}{r'^2 p'^2 \sin^2 \psi};$$

and therefore

$$\frac{a^2 b^2 c^2}{r^2 p^2} = \frac{(a^2 - R^2)(b^2 - R^2)(c^2 - R^2)}{(r^2 - R^2)(p^2 - R^2)}; \text{ but } \frac{p^2}{R^2 - p^2} = \cot^2 \phi;$$

whence, finally, $\qquad \dfrac{R^2}{r^2} - 1 = \dfrac{(a^2 - R^2)(b^2 - R^2)(c - R^2)}{a^2 b^2 c^2} \cot^2 \phi.$

The expression given in this example for $\dfrac{dv}{dt}$ is due to Poinsot.

20. Determine the differential equation of the *herpolhode* (*see* Ex. 7).

If ρ and v be the polar coordinates of the point of contact of the momental ellipsoid with the invariable plane, the origin being the foot of the perpendicular on it from the fixed point, we have

$$\rho^2 = r^2 - p^2,$$

where $\qquad r = \sqrt{\dfrac{K}{S}}\, \omega, \quad p = \dfrac{\sqrt{KS}}{H}$ (Art. 264), and $\dfrac{dv}{dt} = \dfrac{S}{H}\left(1 + \mu \dfrac{p^2}{\rho^2}\right),$

where $\qquad \mu = \dfrac{(AS - H^2)(BS - H^2)(CS - H^2)}{ABCS^3}$ (Ex. 19).

If we express ω in terms of ρ, on substituting in equation (24), Art. 269, we get

$$\frac{d\rho}{dv} = \frac{KH}{S^2} \cdot \frac{\rho}{\rho^2 + \mu p^2} \sqrt{\left\{ \left[\lambda_1 - \frac{S}{K}(p^2 + \rho^2) \right] \left[\lambda_2 - \frac{S}{K}(p^2 + \rho^2) \right] \left[\lambda_3 - \frac{S}{K}(p^2 + \rho^2) \right] \right\}}.$$

274. **Impact.**—When two smooth bodies moving in any way collide, the results of the impact are obtained in a manner precisely similar to that employed in Article 243.

When the motion is wholly unrestricted there are thirteen unknown quantities and thirteen equations.

If λ, μ, ν be the angles made by the common normal at the point of contact with axes fixed in space; R the whole impulse of the mutual normal action during the *first period of impact*; p and p' the perpendiculars on its line of action from the centres of inertia of the two bodies; α, β, γ, α', β', γ' the angles made with the principal axes of the bodies by the axes of the couples produced by R round these points; twelve of the equations mentioned above are

$$\left.\begin{array}{lll} \mathfrak{M}\mathfrak{u} = R\cos\lambda, & & A\varpi_1 = Rp\cos\alpha \\ \mathfrak{M}\mathfrak{v} = R\cos\mu, & & B\varpi_2 = Rp\cos\beta \\ \mathfrak{M}\mathfrak{w} = R\cos\nu, & & C\varpi_3 = Rp\cos\gamma \\ \mathfrak{M}'\mathfrak{u}' = -R\cos\lambda, & & A'\varpi_1' = -Rp'\cos\alpha' \\ \mathfrak{M}'\mathfrak{v}' = -R\cos\mu, & & B'\varpi_2' = -Rp'\cos\beta' \\ \mathfrak{M}'\mathfrak{w}' = -R\cos\nu, & & C'\varpi_3' = -Rp'\cos\gamma' \end{array}\right\}, \quad (46)$$

where $\mathfrak{u}$, &c., are the changes of the components of the velocity of the centre of inertia of the first body, parallel to axes fixed in space, produced during the first period of impact; ϖ_1, &c., the changes of the angular velocities round the principal axes through the centre of inertia produced during the same period; and $\mathfrak{u}'$, &c., have similar significations for the second body.

At the end of the first period the actual components of the velocity of the centre of inertia of the first body are $\mathfrak{u} + \dot{x}_0$, &c., where $\dot{x}_0$ represents the component of this

velocity immediately before the impact. In like manner, $\varpi_1 + \Omega_1$ is the actual angular velocity round the first principal axis. We can then write down, in terms of $\mathfrak{u} + \dot{x}_0$, $\varpi_1 + \Omega_1$, &c., the relative normal velocity of the points of the two bodies which are in contact. Equating this relative normal velocity to zero gives a thirteenth equation; so that $\mathfrak{u}$, ϖ_1, &c., become completely known.

If $\dot{x}$ be the component of the final velocity of the centre of inertia of the first body at the end of *the second period of impact*, and ω_1 the final angular velocity round the first principal axis, &c., the values of the velocities at the end of the impact can now be determined, by aid of the following equations—

$$\left.\begin{aligned} \dot{x} - \dot{x}_0 &= (1+e)\,\mathfrak{u}, \quad \omega_1 - \Omega_1 = (1+e)\,\varpi_1, \text{ \&c.,} \\ \dot{x}' - \dot{x}_0' &= (1+e)\,\mathfrak{u}, \quad \omega_1' - \Omega_1' = (1+e)\,\varpi_1', \text{ \&c.} \end{aligned}\right\}. \quad (47)$$

Since the positions of the two bodies are not sensibly altered during the whole period of impact, it is to be observed that throughout this period any lines fixed in either body coincide with lines fixed in space.

275. **Impulsive Friction.**—When collision takes place between two rough surfaces we can investigate the motion according to the principles laid down in Article 247.

The elementary impulse dF of friction, at each instant of the impact, is to be resolved into two components, dP and dQ, along two tangents through the point of contact at right angles to each other. At any instant during the impact, P represents the entire impulse in a given direction due to the action of friction up to that instant. A similar remark applies to Q, and R is the corresponding impulse due to the normal reaction.

If at any instant during the impact u, v, w be the components, along the two tangents and the normal, of the *relative* tangential and normal velocities of the points of the two surfaces which are in contact, u, v, w can be expressed in terms of the velocities of the two centres of inertia and of the angular velocities of the bodies at that instant; they are

therefore linear functions of P, Q, R. If slipping take place its direction coincides with that of the elementary impulse of friction, and therefore

$$\frac{dP}{dQ} = \frac{u}{v}; \quad \text{also} \quad \sqrt{(dP^2 + dQ^2)} = \mu dR.$$

Initially R is zero, and therefore so likewise are P and Q, except the colliding surfaces be perfectly rough. When $R = R_1$, at the end of the first period of impact, $w = 0$; and if R_2 be the value of R at the end of the whole impact,

$$R_2 = (1 + e) R_1.$$

If the surfaces which collide be *perfectly rough*, the equations $u = 0$, $v = 0$, $w = 0$ enable us to determine P_1, Q_1, R_1. Knowing the value of R_2 we can find P_2 and Q_2 from the equations $u = 0$, $v = 0$, which hold good throughout the impact.

If the bodies *slip on each other in the same direction* during the whole of the impact, the direction of dF is constant, and we may take $dQ = 0$, $dP = \mu dR$. Hence $P_1 = \mu R_1$, $Q_1 = 0$; these equations, with $w = 0$, determine R_1; then

$$P_2 = \mu (1 + e) R_1.$$

276. **Collision of Rough Spheres.**—If a *homogeneous sphere* impinge against a *fixed* surface, or *two homogeneous spheres* collide with each other, by taking as axes of x, y, z parallels to two tangents and the normal at the point of contact, *at any instant during the impact*, we have

$$\left.\begin{array}{lll}
\dot{x} = \dot{x}_0 - \dfrac{P}{\mathfrak{M}}, & \dot{y} = \dot{y}_0 - \dfrac{Q}{\mathfrak{M}}, & \dot{z} = \dot{z}_0 - \dfrac{R}{\mathfrak{M}}, \\[2ex]
\dot{x}' = \dot{x}_0' + \dfrac{P}{\mathfrak{M}'}, & \dot{y}' = \dot{y}_0' + \dfrac{Q}{\mathfrak{M}'}, & \dot{z}' = \dot{z}_0' + \dfrac{R}{\mathfrak{M}'} \\[2ex]
\omega_1 = \Omega_1 + \dfrac{aQ}{I}, & \omega_2 = \Omega_2 - \dfrac{aP}{I}, & \omega_3 = \Omega_3, \\[2ex]
\omega_1' = \Omega_1' + \dfrac{a'Q}{I'}, & \omega_2' = \Omega_2' - \dfrac{a'P}{I'}, & \omega_3' = \Omega_3',
\end{array}\right\} \quad (48)$$

where a, $\mathfrak{M}$, and I are the radius, mass, and moment of inertia

round its diameter, of the first sphere: $\dot{\bar{x}}$, &c., the components of the velocity of its centre, ω_1, &c., the components of its angular velocity: $\dot{\bar{x}}_0$, &c., and Ω_1, &c., the values of these components immediately before the impact: and a', $\mathfrak{M}'$, &c., have similar significations for the second sphere.

At the same instant the velocities of the points of the spheres which are in contact are given by the equations

$$\dot{x} = \dot{\bar{x}} - \eta\omega_3 + \zeta\omega_2, \quad \dot{x}' = \dot{\bar{x}}' - \eta'\omega_3' + \zeta'\omega_2', \text{ \&c.,}$$

where ξ, η, ζ; ξ', η', ζ', are the coordinates of the point of contact relative to the centres of the two spheres; and the relative velocities u, v, w are then determined by the equations $u = \dot{x} - \dot{x}'$, &c.

Hence, since $\xi = 0$, $\eta = 0$, $\zeta = a$, $\xi' = 0$, $\eta' = 0$, $\zeta' = -a'$, substituting for $\dot{\bar{x}}$, ω_1, &c., their values given by (48), we have

$$u = \dot{\bar{x}}_0 - \dot{\bar{x}}_0' + a\Omega_2 + a'\Omega_2' - \left(\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{M}'} + \frac{a^2}{I} + \frac{a'^2}{I'}\right)P,$$

$$v = \dot{\bar{y}}_0 - \dot{\bar{y}}_0' - a\Omega_1 - a'\Omega_1' - \left(\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{M}'} + \frac{a^2}{I} + \frac{a'^2}{I'}\right)Q,$$

$$w = \dot{\bar{z}}_0 - \dot{\bar{z}}_0' - \left(\frac{1}{\mathfrak{M}} + \frac{1}{\mathfrak{M}'}\right)R,$$

that is $\quad u = u_0 - lP, \quad v = v_0 - lQ, \quad w = w_0 - nR,$
where l and n are constants.

Hence $\quad du = -ldP, \quad dv = -ldQ, \quad \dfrac{du}{dv} = \dfrac{dP}{dQ};$

but, if there be slipping, we have, Art. 275,

$\dfrac{dP}{dQ} = \dfrac{u}{v}$; therefore $\dfrac{du}{dv} = \dfrac{u}{v}$, and accordingly $\dfrac{u}{v}$ is constant, or *the direction of slipping is invariable throughout the impact.* Moreover, if either u or v vanish so must the other. All slipping then ceases, and cannot recommence, as u and v are independent of R, and friction cannot initiate slipping. Since, in the present case, R_1 is independent of P and Q, if there be no slipping at the end of the impact the result is the same as if there had been no slipping at all.

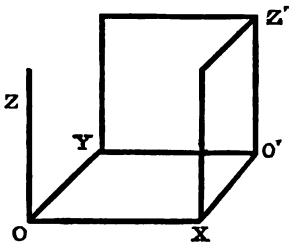
Hence, in all cases, either the impulse of friction is a maximum, and the direction of slipping the same throughout the impact, or else the surfaces may be regarded as perfectly rough.

If the problem be solved on the latter hypothesis, and the value resulting for $\sqrt{(P_2^2 + Q_2^2)}$ does not exceed $\mu(1 + e) R_1$, the solution is correct. If $\sqrt{(P_2^2 + Q_2^2)}$ be greater than $\mu(1 + e) R_1$, slipping takes place in the same direction throughout the impact.

EXAMPLES.

1. A sphere, having no original velocity of rotation, impinges successively against two perfectly rough vertical walls at right angles to each other, the points of impact being so near the intersection of the walls that the action of gravity between the two impacts may be neglected; determine the magnitude and direction of the velocity of the centre of the sphere after the second impact.

Take as axes of ξ, η, ζ, three lines through the centre of the sphere parallel to the intersections of the walls with the plane of the horizon and with each other. Let the components of the velocity of the sphere before the first impact be V_1, V_2, V_3, and let the sphere impinge first against the plane $YO'Z'$.



At the first impact the coordinates ξ, η, ζ of the point of contact are given by the equations

$$\xi = 0, \quad \eta = a, \quad \zeta = 0.$$

Hence, Art. 255, (3),

$$\dot{x} = \dot{\bar{x}} - a\omega_3, \quad \dot{y} = \dot{\bar{y}}, \quad \dot{z} = \dot{\bar{z}} + a\omega_1,$$

where x, y, z are the coordinates of the point of contact referred to axes meeting at O.

Again, $\underset{\cdot}{X}$, $\underset{\cdot}{Z}$ and R being the entire impulses due to friction and to the normal reaction up to the end of the first period of the collision, we have

$$m\dot{\bar{x}} = mV_1 + \underset{\cdot}{X}, \quad m\dot{\bar{y}} = mV_2 + R, \quad m\dot{\bar{z}} = mV_3 + \underset{\cdot}{Z},$$

$$\tfrac{2}{5}ma^2\omega_1 = a\underset{\cdot}{Z}, \quad \omega_2 = 0, \quad \tfrac{2}{5}ma^2\omega_3 = -a\underset{\cdot}{X}, \quad \dot{x} = 0, \quad \dot{y} = 0, \quad \dot{z} = 0.$$

Hence $\quad \dot{\bar{x}} = \frac{5}{7}V_1, \quad \dot{\bar{y}} = 0, \quad \dot{\bar{z}} = \frac{5}{7}V_3, \quad a\omega_1 = -\frac{5}{7}V_3, \quad a\omega_2 = 0, \quad a\omega_3 = \frac{5}{7}V_1.$

At the end of the second period of the collision five of the six velocities above remain unaltered, but $\dot{\bar{y}}$ becomes $-eV_2$.

At the second impact the coordinates of the point of contact are

$$\xi = a, \quad \eta = 0, \quad \zeta = 0.$$

Hence, $\quad \dot{x} = \dot{\bar{x}}, \quad \dot{y} = \dot{\bar{y}} + a\omega_3, \quad \dot{z} = \dot{\bar{z}} - a\omega_2.$

Also, as has been proved above,

$$\dot{x}_0 = \tfrac{5}{7} V_1, \quad \dot{y}_0 = -eV_2, \quad \dot{z}_0 = \tfrac{5}{7} V_3,$$

$$a\Omega_1 = -\tfrac{5}{7} V_3, \quad a\Omega_2 = 0, \quad a\Omega_3 = \tfrac{5}{7} V_1.$$

Proceeding as before, since $\dot{y} = 0$, $\dot{z} = 0$, we obtain

$$a\omega_1 = -\tfrac{5}{7} V_3, \quad a\omega_2 = (\tfrac{5}{7})^2 V_3, \quad a\omega_3 = \tfrac{5}{7} e V_2 + \tfrac{2}{7} \cdot \tfrac{5}{7} V_1,$$

$$\dot{x} = -\tfrac{5}{7} e V_1, \quad \dot{y} = -\tfrac{5}{7} e V_2 - \tfrac{2}{7} \cdot \tfrac{5}{7} V_1, \quad \dot{z} = (\tfrac{5}{7})^2 V_3.$$

2. In the last example, if the walls be not perfectly rough, determine the final velocities of translation and rotation.

We first treat the question as before, and obtain, as in the last example, at the first impact,

$$\frac{1}{m^2}(\underset{\cdot}{X}^2 + \underset{\cdot}{Z}^2) = \tfrac{4}{49}(V_1^2 + V_3^2); \text{ if then } \mu(1+e)V_2 > \tfrac{2}{7}\sqrt{(V_1^2 + V_3^2)}$$

we may assume there is no slipping; but if this condition is not fulfilled slipping takes place, and the maximum amount of friction is exerted. In this case at the end of the first impact,

$$\dot{x} = V_1 - \mu(1+e)V_2 \cos\alpha, \quad \dot{y} = -eV_2, \quad \dot{z} = V_3 - \mu(1+e)V_2 \sin\alpha,$$

$$a\omega_1 = -\tfrac{5}{2}\mu(1+e)V_2 \sin\alpha, \quad \omega_2 = 0, \quad a\omega_3 = \tfrac{5}{2}\mu(1+e)V_2 \cos\alpha,$$

where $\tan\alpha = \dfrac{V_3}{V_1}$.

The values of $\dot{x}$, &c., ω_1, &c., at the end of the first impact are the values of $\dot{x}_0$, &c., Ω_1, &c., at the second impact. If slipping takes place during the whole of the second impact, we have finally

$$\dot{x} = -e\{V_1 - \mu(1+e)V_2 \cos\alpha\},$$

$$\dot{y} = -eV_2 - \mu(1+e)\{V_1 - \mu(1+e)V_2 \cos\alpha\}\cos\beta,$$

$$\dot{z} = V_3 - \mu(1+e)\{V_2 \sin\alpha + V_1 \sin\beta - \mu(1+e)V_2 \cos\alpha \sin\beta\},$$

$$a\varpi_1 = 0, \quad a\varpi_2 = \tfrac{5}{2}\mu(1+e)\{V_1 - \mu(1+e)V_2 \cos\alpha\}\sin\beta,$$

$$a\varpi_3 = -\tfrac{5}{2}\mu(1+e)\{V_1 - \mu(1+e)V_2 \cos\alpha\}\cos\beta,$$

$$a\omega_1 = -\tfrac{5}{2}\mu(1+e)V_2 \sin\alpha,$$

$$a\omega_2 = \tfrac{5}{2}\mu(1+e)\{V_1 - \mu(1+e)V_2 \cos\alpha\}\sin\beta,$$

$$a\omega_3 = \tfrac{5}{2}\mu(1+e)\{V_2 \cos\alpha[1 + \mu(1+e)\cos\beta] - V_1 \cos\beta\},$$

where $$\tan\beta = \frac{2\{V_3 - \mu(1+e)V_2 \sin\alpha\}}{5\mu(1+e)V_2 \cos\alpha - 2eV_2}.$$

277. **Equations of Motion referred to Body-Axes.**—If u, v, w be the components of the velocity of the centre of inertia at any instant in the directions of the principal axes of the body at that point, the accelerations of the centre of inertia in these directions are, by Art. 257,

$$\frac{du}{dt} - v\omega_3 + w\omega_2, \text{ \&c.};$$

whence, if ΣX, ΣY, ΣZ be the sums of the components of the applied forces at any instant, parallel to the principal axes through the centre of inertia, its equations of motion are

$$\left.\begin{aligned} \mathfrak{M}\left\{\frac{du}{dt} - v\omega_3 + w\omega_2\right\} &= \Sigma X \\ \mathfrak{M}\left\{\frac{dv}{dt} - w\omega_1 + u\omega_3\right\} &= \Sigma Y \\ \mathfrak{M}\left\{\frac{dw}{dt} - u\omega_2 + v\omega_1\right\} &= \Sigma Z \end{aligned}\right\}. \tag{49}$$

278. **Motion consisting of Successive Rotations.**—If the whole motion of a body consist of successive rotations (not necessarily executed round lines passing through the same point), the *vis viva* at any time is $I\omega^2$, where I is the moment of inertia, and ω the angular velocity round the instantaneous axis; hence

$$I\omega^2 = 2\Sigma\int(Xdx + Ydy + Zdz) + \text{const.};$$

therefore $$\frac{d}{dt}(I\omega^2) = 2\Sigma\left(X\frac{dx}{dt} + Y\frac{dy}{dt} + Z\frac{dz}{dt}\right).$$

Let p be the perpendicular on the instantaneous axis from the point x, y, z; and let the direction of the motion of this point make angles α, β, γ with the axes; then

$$\frac{dx}{dt} = p\omega\cos\alpha,\quad \frac{dy}{dt} = p\omega\cos\beta,\quad \frac{dz}{dt} = p\omega\cos\gamma;$$

whence $\frac{d}{dt}(I\omega^2) = 2\omega\Sigma p\,(X\cos\alpha + Y\cos\beta + Z\cos\gamma;$

but $$(X\cos\alpha + Y\cos\beta + Z\cos\gamma)p$$

is the moment of the force applied at the point x, y, z round the instantaneous axis. Hence, if J be the moment of the entire system of applied forces round the instantaneous axis, we get

$$\frac{d}{dt}(I\omega^2) = 2\omega J,$$

or $$\frac{1}{\omega}\frac{d}{dt}(I\omega^2) = 2J. \qquad (50)$$

If the body be such that the moments of inertia round the different instantaneous axes are equal, this equation takes the simple form

$$I\frac{d\omega}{dt} = J. \qquad (51)$$

EXAMPLES.

1. A homogeneous sphere, having an initial angular velocity round a horizontal axis, is projected along a rough horizontal plane: determine the motion, neglecting the couple of rolling friction. (Jellett, *Theory of Friction*, Chap. V.).

The axes being three mutually perpendicular lines through the centre, whose directions are fixed in space, equations (17) of Art. 267 become for a homogeneous sphere whose radius is r,

$$\tfrac{2}{5}\mathfrak{M}r^2\frac{d\omega_1}{dt} = L, \quad \tfrac{2}{5}\mathfrak{M}r^2\frac{d\omega_2}{dt} = M, \quad \tfrac{2}{5}\mathfrak{M}r^2\frac{d\omega_3}{dt} = N,$$

where L, M, N are the moments of the forces round the axes.

Let X and Y be the components of friction along two horizontal axes, x and y the coordinates of the centre of the sphere, u and v the components of the velocity of its point of contact with the rough plane; then, by Art. 255,

$$u = \frac{dx}{dt} - r\omega_2, \quad v = \frac{dy}{dt} + r\omega_1.$$

The equations of motion are

$$\mathfrak{M}\frac{d^2x}{dt^2} = X, \quad \mathfrak{M}\frac{d^2y}{dt^2} = Y, \quad \tfrac{2}{5}\mathfrak{M}r^2\frac{d\omega_1}{dt} = rY, \quad \tfrac{2}{5}\mathfrak{M}r^2\frac{d\omega_2}{dt} = -rX.$$

Combining the first and last of these, we have

$$\frac{du}{dt}=\frac{d^2x}{dt^2}-r\frac{d\omega_2}{dt}=\frac{2}{7}\frac{X}{\mathfrak{M}}.$$

In like manner

$$\frac{dv}{dt}=\frac{2}{7}\frac{Y}{\mathfrak{M}},\quad \text{whence } \frac{du}{dv}=\frac{X}{Y}.$$

Now if there be slipping,

$$X=-\mu\mathfrak{M}g\frac{u}{\sqrt{(u^2+v^2)}},\quad Y=-\mu\mathfrak{M}g\frac{v}{\sqrt{(u^2+v^2)}};\quad \therefore\ \frac{X}{Y}=\frac{u}{v};$$

whence $\frac{u}{v}=\text{constant}=\cot\alpha=\frac{V_1-r\Omega_2}{V_2+r\Omega_1}$, where V_1, V_2, Ω_1, Ω_2 are the initial values of $\frac{dx}{dt}$, $\frac{dy}{dt}$, ω_1, and ω_2. Hence

$$X=-\mu\mathfrak{M}g\cos\alpha,\quad Y=-\mu\mathfrak{M}g\sin\alpha.$$

These are the components of a constant force in a fixed direction. Hence in general the centre of the sphere describes a parabola. If, however, the initial axis of rotation be perpendicular to the direction of the initial motion of the centre, *i.e.* if $V_1\Omega_1+V_2\Omega_2=0$, the centre of the sphere continues to move in the direction of its initial motion.

Substituting the values of X and Y in the equations of motion, we find that slipping ceases along the axis of x when $t=\frac{2(V_1-r\Omega_2)}{7\mu g\cos\alpha}$, and along the axis of y when $t=\frac{2(V_2+r\Omega_1)}{7\mu g\sin\alpha}$; but $\frac{V_2+r\Omega_1}{\sin\alpha}=\frac{V_1-r\Omega_2}{\cos\alpha}$, hence, slipping along each axis ceases at the same time, t_0, where

$$t_0=\frac{2}{7}\frac{\sqrt{\{(V_1-r\Omega_2)^2+(V_2+r\Omega_1)^2\}}}{\mu g}.$$

After pure rolling begins it will continue, since the values which X and Y must take in order to maintain it are zero; the components of the velocity of the centre are then given by the equations

$$\frac{dx}{dt}=\frac{5V_1+2r\Omega_2}{7},\quad \frac{dy}{dt}=\frac{5V_2-2r\Omega_1}{7},$$

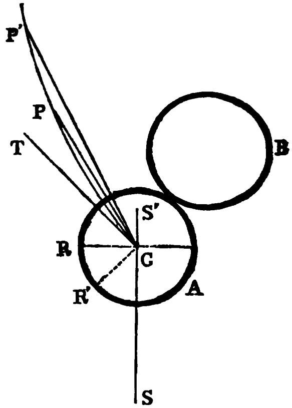
and the remainder of the path is a straight line. If β be the angle which the final line of motion of the ball makes with the axis of x,

$$\tan\beta=\frac{5V_2-2r\Omega_1}{5V_1+2r\Omega_2}.$$

The result here obtained, that the centre of the sphere may describe a parabola, enables us to explain a well-known phenomenon in billiards, which may be stated as follows :—

The angle at which the striker's ball goes off the ball aimed at, in order to make a cannon, seems to be less according as the distance of the third ball is greater.

Let A be the striker's ball, B the ball first struck by A. If A be struck in the ordinary way, without side, it will have, when it strikes B, a rotation round a horizontal axis CR at right angles to SC, the line of original motion. This rotation will continue round the same axis after impact. Suppose the motion of translation after impact to be in the direction CT, then CR not coinciding with CR', the horizontal line perpendicular to CT, the path described by the ball is the parabolic arc CPP'. Hence, if P' be more remote than P, PCS' is greater than $P'CS'$.



It is well known that a skilful billiard player can make a ball describe a very marked curve. This is done by an impulse having a *vertical component* which imparts a rotation round the line of original translation of the centre.

If the original impulse be horizontal it produces no moment round a line through the centre parallel to itself, and this latter being the original line of translation of the centre, there can be no rotation round it; hence in this case the ball must move in a straight line.

2. A sphere rolls along a rough horizontal plane. Taking into account the couple of rolling friction determine the forces brought into play and the path described, the motion being pure rolling.

The equations of motion are

$$\mathfrak{M}\frac{d^2x}{dt^2}=X, \quad \mathfrak{M}\frac{d^2y}{dt^2}=Y,$$

$$\tfrac{2}{5}\mathfrak{M}r^2\frac{d\omega_1}{dt}=rY-f\mathfrak{M}g\frac{\omega_1}{\omega}, \quad \tfrac{2}{5}\mathfrak{M}r^2\frac{d\omega_2}{dt}=-rX-f\mathfrak{M}g\frac{\omega_2}{\omega}, \text{ where } \omega=\sqrt{\omega_1^2+\omega_2^2},$$

with the conditions $$\frac{dx}{dt}-r\omega_2=0, \quad \frac{dy}{dt}+r\omega_1=0,$$

whence $$X=-\tfrac{5}{7}\frac{f}{r}\mathfrak{M}g\frac{\omega_2}{\omega}, \quad Y=\tfrac{5}{7}\frac{f}{r}\mathfrak{M}g\frac{\omega_1}{\omega}, \quad \frac{Y}{X}=\frac{-\omega_1}{\omega_2}=\frac{\dot{y}}{\dot{x}}.$$

Hence the path is a straight line. Multiplying the third equation of motion by ω_1, the fourth by ω_2, and adding, we have

$$\tfrac{2}{5}\mathfrak{M}r^2\omega\frac{d\omega}{dt}=-\tfrac{2}{7}f\mathfrak{M}g\omega,$$

whence $$r\omega=-\tfrac{5}{7}\frac{fg}{\ }t+r\Omega,$$

Ω being the initial angular velocity.

The sphere will come to rest when $t = \dfrac{7r^2\Omega}{5fg}$.

Again, multiplying the third equation of motion by ω_2, the fourth by ω_1, and subtracting, we have $\omega_2 d\omega_1 - \omega_1 d\omega_2 = 0$. Hence $\dfrac{\omega_1}{\omega_2} = \text{constant} = -\tan\alpha$, where α is the angle which the path makes with the axis of x.

3. A sphere is projected obliquely down a rough inclined plane, the motion being pure rolling; determine the friction brought into play, and the path, neglecting the couple of rolling friction.

Take as axis of x the intersection of the inclined plane with a vertical plane at right angles thereto.

The equations of motion are

$$\mathfrak{M}\frac{d^2x}{dt^2} = X + \mathfrak{M}g\sin i,\quad \mathfrak{M}\frac{d^2y}{dt^2} = Y,\quad \tfrac{2}{5}\mathfrak{M}r^2\frac{d\omega_1}{dt} = rY,\quad \tfrac{2}{5}\mathfrak{M}r^2\frac{d\omega_2}{dt} = -rX,$$

with the conditions $\qquad \dfrac{dx}{dt} - r\omega_2 = 0,\quad \dfrac{dy}{dt} + r\omega_1 = 0;$

whence $\qquad Y = 0,\quad X = -\tfrac{2}{7}\mathfrak{M}g\sin i.$

The whole force, therefore, is $\tfrac{5}{7}\mathfrak{M}g\sin i$ parallel to the axis of x, and the centre of the sphere being acted on by a constant force parallel to a fixed direction, describes a parabola. Also, since

$$\omega_1\frac{dx}{dt} + \omega_2\frac{dy}{dt} = 0,$$

the instantaneous axis of rotation is at right angles to the tangent to the path of the point of contact on the inclined plane. This is otherwise immediately obvious, since the motion is pure rolling.

279. Equations of Motion of a Solid of Revolution.—If one point O of a rigid body be fixed in space, and two of the principal moments of inertia at the point be equal, the equations of motion of the body can be expressed in a comparatively simple form.

Let OC (Art. 258) be the axis of revolution of the momental ellipsoid of the body, and A and C its principal moments of inertia at O, then, by considering the motion of a point situated on OC, it is plain that the angular velocity of the body round an axis OS perpendicular to OC in the plane ZOC is $\dot\psi\sin\theta$, and the moment of momentum round OS is therefore $A\dot\psi\sin\theta$. Hence

$$H_z = A\dot\psi\sin^2\theta + C\omega_3\cos\theta. \qquad (52)$$

Again, the angular velocity of the body round OE perpendicular to the plane ZOC is $\dot{\theta}$, and therefore we have

$$2T = A(\dot{\psi}^2 \sin^2\theta + \dot{\theta}^2) + C\omega_3^2. \qquad (53)$$

If G_x, G_y, G_z be the moments round the space-axes of the applied forces, and Υ the force function, we have then, as the three equations of motion of the body,

$$\left.\begin{aligned} &\frac{d}{dt}(A\dot{\psi}\sin^2\theta + C\omega_3\cos\theta) = G_z \\ &A(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2) + C\omega_3^2 = 2\Upsilon + \text{constant} \\ &C\frac{d\omega_3}{dt} = N = \sin\theta(G_x\cos\psi + G_y\sin\psi) + G_z\cos\theta \end{aligned}\right\}. \qquad (54)$$

We may if we please substitute $\dot{\phi} + \dot{\psi}\cos\theta$ for ω_3 in (52), (53), and (54).

Equations similar to (54) hold good for a free body if two of its principal moments of inertia at its centre of inertia be equal. In this case OZ is a parallel through the centre of inertia to a line fixed in space.

Examples.

1. A homogeneous solid of revolution terminating in a cone is placed with the vertex of the cone on a perfectly rough horizontal plane, the initial conditions being given, find the equations of motion.

Here the vertex of the cone is the fixed point O; and if a vertical line through O be taken as the space-axis OZ, since gravity is the only force, G_z and N are each zero. Then by (54) we have $\omega_3 = \text{constant} = n$, and therefore the first two of equations (54) become

$$A\dot{\psi}\sin^2\theta + Cn\cos\theta = K, \quad A(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2) + Cn^2 = E - 2mgh\cos\theta, \quad (a)$$

where h is the distance of the centre of inertia from O.

If we take a point P on OC the axis of revolution at a distance l from O such that $mhl = A$, this point P is the centre of oscillation of the body for an axis perpendicular to ZOC. Assuming $Kl = Cna$, and $(E - Cn^2)l = 2mghb$, we have, then, to determine the motion of P the equations

$$\left.\begin{aligned} mhl^2\dot{\psi}\sin^2\theta &= Cn(a - l\cos\theta) \\ l^2(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2) &= 2g(b - l\cos\theta) \end{aligned}\right\}. \qquad (b)$$

2. Give a geometrical construction for the velocity of P in any position.

Take on OZ two points D and F such that $OD = a$, $OF = b$, and let χ be the angle which PD makes with the plane of the horizon, then at any instant the velocity of P is that due to the depth of P below the horizontal plane through F, and the component of this velocity perpendicular to the plane ZOC is

$$\frac{Cn}{mh}\tan\chi.$$

3. Show that the motion of the axis of revolution may be represented by that of the conjugate line (Art. 270) in a body not acted on by any force.

The equations of motion of OC in Ex. 1 are of the same form as those of the conjugate line in Ex. 3, Art. 270, and by properly determining the disposable constants in the latter they may be made identical with the former.

This theorem was first given by Jacobi, but the mode of investigation here adopted is due to Dr. Routh.

4. Determine the limits of the inclination of the axis of revolution to the vertical.

Eliminating $\dot{\psi}$ from equations (b) of Ex. 1 we obtain

$$l^2\dot{\theta}^2 = 2g(b - l\cos\theta) - \left(\frac{Cn}{mh}\right)^2\left(\frac{a - l\cos\theta}{l\sin\theta}\right)^2. \qquad (a)$$

Now when θ attains its limiting value, $\dot{\theta} = 0$, and therefore to determine the limiting values of θ we have the equation

$$2A^2g(b - l\cos\theta)(1 - \cos^2\theta) - C^2n^2(a - l\cos\theta)^2 = F(\cos\theta) = 0. \qquad (b)$$

From equation (a) it is plain that F cannot be negative for any value of θ attained in the actual motion of the body. Hence, if i be the initial value of θ, F is positive or zero when $\cos\theta = \cos i$. Again, it is easy to see that F is negative for $\cos\theta = -1$, or $\cos\theta = 1$, and positive for $\cos\theta = \infty$. We conclude that the equation $F(\cos\theta) = 0$ has three real roots, two between -1 and $+1$, and one between $+1$ and ∞. This last root is an impossible value for $\cos\theta$. In general, then, the angle θ oscillates between two limiting values α_1 and α_2, one less and the other greater than the initial value i. That this oscillation should be possible, it is necessary, however, that F should vanish before any point of the body above the point of support comes into contact with the horizontal plane. If β be the value of θ for the position in which such contact takes place, in order that an oscillation should be possible, $F(\cos\beta)$ must be negative, and therefore

$$C^2n^2(a - l\cos\beta)^2 > 2A^2g(b - l\cos\beta)\sin^2\beta. \qquad (c)$$

In terms of the original constants, K and E, this condition becomes

$$(A\sin^2\beta + C\cos^2\beta)\,Cn^2 - 2KCn\cos\beta > A(E - 2mgh\cos\beta)\sin^2\beta - K^2. \qquad (d)$$

5. Show that the character of the oscillating motion depends on the relative magnitudes of a and b.

If in equation (*a*) Ex. 4, we make $l\cos\theta = a$, we get $l^2\dot\theta^2 = 2g(b-a)$. If a be less than b this gives the value of $\dot\theta$ when $\dot\psi = 0$ ((*b*) Ex. 1.) In this case the angular motion of the plane ZOC changes its direction at the point corresponding to $a = l\cos\theta$.

Again, if $a > b$, the relation $a = l\cos\theta$ leads to an imaginary value for $\dot\theta$, and consequently $\dot\psi$ cannot vanish during the motion. Hence in this case the axis OC rotates constantly in one direction round the vertical line OZ.

6. Determine in any particular case of the motion whether a or b is the greater.

If $a > b$, then, by Ex. 1, $\dfrac{K}{Cn} > \dfrac{E - Cn^2}{2mgh}$.

7. If the axis of revolution rotate constantly in one direction round the vertical, and if $\dot\psi_0$ be the value of $\dot\psi$ which corresponds to either the greatest or least value of θ, prove that $\dot\psi_0 < \dfrac{2mgh}{Cn}$.

8. Find the conditions which must be fulfilled in order that the motion of OC should be steady.

In this case, if it can occur, the inclination of OC to the vertical and the angular velocity of the plane ZOC are constant. If we eliminate $\dot\theta$ between the two equations obtained by differentiation from (*a*), Ex. 1, we get

$$A\ddot\theta = A\dot\psi^2\sin\theta\cos\theta - Cn\dot\psi\sin\theta + mgh\sin\theta. \tag{a}$$

Hence if $\ddot\theta = 0$, we have

$$\sin\theta\,(A\dot\psi^2\cos\theta - Cn\dot\psi + mgh) = 0. \tag{b}$$

If $\sin\theta = 0$, we have $\theta = 0$. In this case the axis is vertical throughout the motion. Again, if

$$A\cos\theta\dot\psi^2 - Cn\dot\psi + mgh = 0,$$

we obtain
$$\dot\psi = \frac{Cn \pm \sqrt{(C^2n^2 - 4Amgh\cos\theta)}}{2A\cos\theta}.$$

If then i the initial value of θ fulfil the condition

$$C^2n^2 > 4Amgh\cos i,$$

and if likewise initially $\dot\theta = 0$, and

$$\dot\psi = \frac{Cn \pm \sqrt{(C^2n^2 - 4Amgh\cos i)}}{2A\cos i}, \tag{c}$$

all the successive differential coefficients of $\dot\theta$ and $\dot\psi$ must vanish initially, as readily appears from the expression for $\ddot\theta$ and the first of equations (*a*), Ex. 1, and therefore θ and $\dot\psi$ remain constant, and the motion is steady.

9. Prove that if the motion be not steady initially it cannot become so subsequently.

In order that the motion should become steady it would be necessary that $\dot\theta$ and $\ddot\theta$ should vanish simultaneously. $\dot\theta$ is given in terms of θ by equation (*a*), Ex. 4, which is of the form $k\dot\theta^2\sin^2\theta = F(\cos\theta)$, where k is constant. If we

differentiate this equation, divide both sides by $\dot\theta \sin\theta$, and then make $\dot\theta$ and $\ddot\theta$ each zero, we obtain $F'(\cos\theta) = 0$. Hence if $\dot\theta$ and $\ddot\theta$ vanish together, the equation $F(\cos\theta) = 0$ must have equal roots. Now (Ex. 4),

$$F(\cos\theta) = 2A^2gl(\cos\theta - \cos\alpha_1)(\cos\theta - \cos\alpha_2)(\cos\theta - \lambda),$$

where λ is always greater than 1. Hence if the equation $F(\cos\theta) = 0$ have equal roots, we must have $\alpha_1 = \alpha_2$, and as θ in the actual motion always lies between α_1 and α_2, the double root must be $\cos i$, where i is the initial value of θ. Consequently, if the motion be not steady originally it can never become so.

10. A peg-top is set spinning on a rough plane, determine the motion.

In this case the only initial motion is a rotation round OC, and therefore, if i be the initial value of θ, we have

$$K = Cn\cos i, \quad E - Cn^2 = 2mgh\cos i.$$

Hence $a = b = l\cos i$, and equations (b) Ex. 1, become

$$mhl\dot\psi\sin^2\theta = Cn(\cos i - \cos\theta), \quad l(\dot\psi^2\sin^2\theta + \dot\theta^2) = 2g(\cos i - \cos\theta). \quad (a)$$

The latter equation shows that $\cos i > \cos\theta$, and therefore that $\dot\psi$ has the same sign as n. Hence the rotation of the top round its axis is in the same direction as the rotation of the latter round the vertical through the point of support.

Again, if we put $C^2n^2 = 4\nu mghA = 4\nu m^2gh^2l$, equation ($b$), Ex. 4, becomes

$$(\cos i - \cos\theta)\{1 - 2\nu\cos i + 2\nu\cos\theta - \cos^2\theta\} = 0. \quad (b)$$

Hence $\theta = i$ or i', where i' is determined by the equation $\sin^2 i' = 2\nu(\cos i - \cos i')$, and θ oscillates between its least value i and its greatest value i', provided $i' < \beta$, that is $2\nu > \dfrac{\sin^2\beta}{\cos i - \cos\beta}$. It is plain that the latter equation is what (c) Ex. 4 becomes in the present case.

11. Show that steady motion is impossible in the case of a top, except the initial position of its axis be vertical.

12. Investigate the small oscillations of the axis of a top about its mean inclination to the vertical.

We have seen that if ν or n fulfil the condition given at the end of Ex. 10, $\dot\theta$ vanishes when $\theta = i$, and also when $\theta = i'$. At some position of the axis of revolution intermediate between these two $\ddot\theta = 0$. If ϖ and α be the values of $\dot\psi$ and θ corresponding to this position, we have

$$\left.\begin{aligned} A\varpi^2\cos\alpha - Cn\varpi + mgh &= 0 \\ A\varpi\sin^2\alpha + Cn\cos\alpha &= Cn\cos i \end{aligned}\right\}. \quad (a)$$

The motion would now be steady if $\dot\theta$ were zero. We have seen that this is impossible; but as $\ddot\theta$ is now of the opposite sign to $\dot\theta$ the axis of the top will oscillate about this position, provided θ is small.

To determine these oscillations, let $\dot{\psi} = \varpi + \sigma$, $\theta = \alpha + \epsilon$. Then, σ and ϵ being small, by substituting in (a) Ex. 8, and in the first of equations (a) Ex. 1, we have

$$A\frac{d^2\epsilon}{dt^2} = \{A\varpi^2(\cos^2\alpha - \sin^2\alpha) + (mgh - Cn\varpi)\cos\alpha\}\epsilon + (2A\varpi\sin\alpha\cos\alpha - Cn\sin\alpha)\sigma,$$

$$A\sigma\sin^2\alpha + \sin\alpha\,(2A\varpi\cos\alpha - Cn)\,\epsilon = 0.$$

Substituting for $mgh - Cn\varpi$ its value given by (a), we get from the first of these

$$A\frac{d^2\epsilon}{dt^2} + A\varpi^2\epsilon\sin^2\alpha = (2A\varpi\cos\alpha - Cn)\,\sigma\sin\alpha\,;$$

but from the second we have

$$A\sigma\sin\alpha = (Cn - 2A\varpi\cos\alpha)\,\epsilon,$$

whence, eliminating σ, we obtain

$$\frac{d^2\epsilon}{dt^2} + \frac{A^2\varpi^2\sin^2\alpha + (2A\varpi\cos\alpha - Cn)^2}{A^2}\epsilon = 0,$$

and therefore $\epsilon = j\sin(\mu t + \gamma)$, where j and γ are arbitrary constants, and μ is given by the equation

$$A^2\mu^2 = A^2\varpi^2\sin^2\alpha + (2A\varpi\cos\alpha - Cn)^2.$$

From the expression for $\dot{\theta}^2$ given by (a) Ex. 4, it is easy to see that by properly determining n, we can make θ small throughout the motion, and thus the condition requisite for a small oscillation can be secured.

13. Find the vertical pressure on the plane on which the top is spinning.

If z be the vertical coordinate of the centre of inertia of the top, and P the vertical force exerted on the top by the plane, we have $P = mg + m\ddot{z}$; but

$$\ddot{z} = \frac{1}{2}\frac{d}{dz}\left(\frac{dz}{dt}\right)^2 = \frac{1}{2}h\frac{d}{d\cos\theta}\left(\sin\theta\frac{d\theta}{dt}\right)^2,$$

and from (a) Ex. 4 and Ex. 10, we have

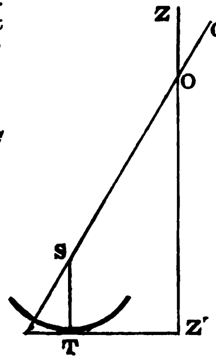
$$\sin^2\theta\,\dot{\theta}^2 = \frac{2g}{l}(\cos i - \cos\theta)\{1 - 2\nu\cos i + 2\nu\cos\theta - \cos^2\theta\}.$$

Hence $$P = mg\left\{1 + \frac{h}{l}(3\cos^2\theta - 2(\cos i + 2\nu)\cos\theta + 4\nu\cos i - 1)\right\}.$$

14. A solid of revolution, having a great angular velocity round its axis, and terminated by a spherical surface of small radius, is placed, with its axis inclined to the vertical, on a rough horizontal plane. The moment of inertia round the axis of revolution being not less than that round an axis perpendicular thereto, and the distance of the centre of inertia from the lower end being considerable, show that after some time the axis of revolution will become vertical. (Jellett, *Theory of Friction*, Chapter VIII.)

Let the axis of z through the centre of inertia O be vertical, and let OC be the axis of revolution, which must pass through S, the centre of the terminating spherical surface. Accordingly the point of contact T lies in the plane ZOS.

The forces acting on the body are gravity, and the resultant of the normal reaction and friction at T. The friction may be resolved into two, one along TZ', the other at right angles to the plane ZOC. Calling this latter component F, and putting



$$TS = a, \quad SO = b, \quad ZOC = \theta,$$

the moments of the applied forces round OZ and OC are respectively

$$Fb \sin\theta, \text{ and } -Fa \sin\theta.$$

Hence, by (54),

$$\frac{d}{dt}\left(A \sin^2\theta\dot{\psi} + C\omega_3 \cos\theta\right) = Fb \sin\theta;$$

also,
$$C\frac{d\omega_3}{dt} = -Fa \sin\theta;$$

therefore
$$\frac{d}{dt}\left(A \sin^2\theta\dot{\psi} + C\omega_3 \cos\theta\right) = -\frac{b}{a}\, C\frac{d\omega_3}{dt}.$$

Hence, putting $n = \dfrac{b}{a}$,

$$A \sin^2\theta\dot{\psi} + C\omega_3 \cos\theta + nC\omega_3 = \text{constant} = C\Omega\,(\cos\theta_0 + n),$$

where Ω and θ_0 are the initial values of ω_3 and θ, since $\dot{\psi} = 0$ initially.

As the force F constantly diminishes the angular velocity, after some time ω_3 must become equal to

$$\Omega\,\frac{n + \cos\theta_0}{n + 1}.$$

When this happens, we have $\theta = 0$. For, substituting in the previous equation the value just obtained for ω_3, we get

$$(1 - \cos\theta)\left\{2A\cos^2\tfrac{1}{2}\theta\dot{\psi} - C\Omega\,\frac{n + \cos\theta_0}{n + 1}\right\} = 0;$$

but as n is greater than 1, $\dot{\psi}$ small as compared with Ω, and C not less than A, the second factor of the above expression cannot vanish, and therefore we must have $\theta = 0$.

The result obtained here may be regarded as holding good in the case of a humming-top.

CHAPTER XII.

ENERGY AND THE GENERAL EQUATIONS OF DYNAMICS.

SECTION I.—*Energy*.

280. **Energy.**—Work and Energy have been defined, Arts. 118 and 129, and the equation of Energy for a rigid body has been obtained by two different methods (Arts. 132 and 200). In the present section we propose to consider the subject in a somewhat more general manner, and to show that on the equation of Energy may be based the whole theory of the action of forces on a connected system.

281. **Equation of Energy.**—If a system of material points be acted on by any forces, we may suppose the constraints and connexions of the system replaced by corresponding forces, and thus regard each point as entirely free. Assuming then the principles which govern the resolution and composition of forces acting at a point, and the relations between force and acceleration, we have

$$m_1 \frac{d^2x_1}{dt^2} = X_1, \quad m_1 \frac{d^2y_1}{dt^2} = Y_1, \quad m_1 \frac{d^2z_1}{dt^2} = Z_1,$$

$$m_2 \frac{d^2x_2}{dt^2} = X_2, \quad m_2 \frac{d^2y_2}{dt^2} = Y_2, \quad m_2 \frac{d^2z_2}{dt^2} = Z_2,$$

$$\&c., \qquad\qquad \&c.,$$

where X_1, Y_1, Z_1, &c., include the components of the forces which replace the constraints, if any, acting at the points $x_1 y_1 z_1$, $x_2 y_2 z_2$, &c.

Multiplying the first equation by dx_1, the second by dy_1, &c., and adding, we have

$$\Sigma m\left(\frac{d^2x}{dt^2}\,dx + \frac{d^2y}{dt^2}\,dy + \frac{d^2z}{dt^2}\,dz\right) = \Sigma\,(Xdx + Ydy + Zdz)\,;$$

or, putting $T = \frac{1}{2}\Sigma mv^2$,

$$\frac{dT}{dt}.dt = \Sigma(Xdx + Ydy + Zdz). \qquad (1)$$

In virtue of this equation, T is called the kinetic energy of the system (Art. 129).

282. **Conservation of Energy.**—*If the mutual forces of a materidl system are independent of the velocities, the system must be conservative* (Art. 124).

To prove this we have to show that, in going from any configuration A to the configuration B, the work done by the forces of the system is independent of the mode of transformation, and depends only on the initial and final configurations.

Suppose that in one mode M of going from A to B more work is done than in another mode N. Let us imagine two systems precisely similar, and let the first, going from A to B by the mode M, be made to bring the other from B to A by the mode N. This will be possible, because the work consumed in going from B to A through N is equal to the work performed in going from A to B through N, since the forces in any particular position are by hypothesis independent of the *directions* in which the points of the system are moving, and therefore each element Xdx of the total work retains the same magnitude, but changes its sign, when the tranformation is reversed. Hence the transformation from A to B through M will bring the second system from B to A through N, and leave an overplus of work. Let us now suppose the second system to go from A to B through M, bringing the first from B to A through N. There will again be an overplus of work. This process may be continually repeated, and thus an inexhaustible supply of work can be obtained from permanent natural causes without any consumption of materials. The whole of experience teaches us that this is impossible. Hence the work done in going from A to B is independent of the mode of transformation.

If a system be acted on by no external forces, the work done by the forces of the system is equal to the change of kinetic energy; whence it appears that the kinetic energy T

in any particular configuration depends only on the values of the variables by which the configuration is indicated, and on the initial state: in other words, we have the equation

$$T - T_0 = \phi\,(x_1, y_1, z_1, x_2, y_2, z_2, \&c.). \qquad (2)$$

It is essential to the validity of this demonstration that the work consumed by the forces of the system, in any transformation, should be equal to the work performed by them in the *same* transformation reversed. If the force acting on any point changes sign with the direction of the motion of that point, the condition of reversibility is not fulfilled. In the case of friction, for instance, so far as it is considered in Mechanics, the forces change sign with the motion, and consume work both in the direct and reverse transformation. The same is true of the resistance of a medium. Again, if the forces are not equal in magnitude when the points occupy the same relative positions, as in the case of the collision of imperfectly elastic bodies, work is apparently consumed without any corresponding increase of potential energy. The experiments of Joule and others have established that in such cases the energy which seems to be lost is really preserved in the form of heat, which may be regarded as kinetic energy resulting from molecular motions not directly sensible. In applying the equation of energy we must, however, remember that in cases such as those mentioned, the conservation of energy, so far as *sensible* motion alone is concerned, does not hold good. On the other hand, if we take into account *every form of energy*, the conservation of energy may be considered as an absolutely universal fact of nature.

Equation (2) may be written

$$T - T_0 = \Upsilon - \Upsilon_0, \qquad (3)$$

where Υ is the force function given by the equation

$$\Upsilon = \Sigma\int(Xdx + Ydy + Zdz).$$

Equation (3) is the same as (12), Art. 200, where it was arrived at in a different manner.

If we put $\Upsilon = -V$, we have

$$T + V = T_0 + V_0. \qquad (4)$$

We can now give an exact definition of potential energy.

The Potential energy of a conservative system in any particular configuration is the amount of work required to bring it to that configuration against the mutual forces of the system in its passage from any chosen configuration.

The principle of the conservation of energy may then be stated thus:—

In any conservative system unacted on by external forces the sum of the kinetic and potential energies is constant.

283. **Of the Ultimate Permanent Forces of Nature.**—In his Paper on the Conservation of Force (*Ueber die Erhaltung der Kraft*, 1847), Helmholtz observes that we must regard the forces of nature as caused by the action of portions of matter on each other, and from a mathematical point of view must consider matter as composed of an infinite number of material points. The ultimate permanent forces of nature must result, therefore, from the action of these material points on each other.

If the conservation of energy hold good for these forces, the mutual forces between two material points must be in the line joining them, and be a function of the distance between them.

This proposition is proved by Helmholtz as follows:—

In this case the kinetic energy of the system composed of the two points is given by the equation

$$T = \int (X dx + Y dy + Z dz + X' dx' + Y' dy' + Z' dz') + c.$$

Since the conservation of energy holds good, T is a function of the relative position of the two points. Again, as they are points, all directions must be supposed indifferent as regards either of them considered alone. Hence their relative position must depend solely on the distance between them, and T is therefore a function of this distance r, or $T = \phi(r)$.

Equating the two expressions for T, and differentiating, we have

$$X = \phi'(r)\frac{dr}{dx}, \quad Y = \phi'(r)\frac{dr}{dy}, \quad Z = \phi'(r)\frac{dr}{dz},$$

$$X' = \phi'(r)\frac{dr}{dx'}, \quad Y' = \phi'(r)\frac{dr}{dy'}, \quad Z' = \phi'(r)\frac{dr}{dz'};$$

$$\text{or} \quad X = \phi'(r)\frac{x-x'}{r}, \quad Y = \phi'(r)\frac{y-y'}{r}, \quad Z = \phi'(r)\frac{z-z'}{r},$$

$$X' = \phi'(r)\frac{x'-x}{r}, \quad Y' = \phi'(r)\frac{y'-y}{r}, \quad Z' = \phi'(r)\frac{z'-z}{r}.$$

Hence the point xyz is acted on by a force $\phi'(r)$ in the direction of the line joining $x'y'z'$ to xyz; and the latter point is acted on by an equal force in the opposite direction.

Conversely it is easy to see that, if two material points acted on each other with a force depending as regards magnitude on their mutual distance, but *not in the direction of the line joining them*, they would be capable of producing in each other an ever-increasing velocity, and of thus generating an unlimited amount of energy.

In order to bring about this result we have only to suppose the points connected by a rigid rod. The whole system would then be acted on by a constant couple.

284. **Forces which appear in the Equation of Energy.**—For any system entirely free we have obtained the equation $dT = \Sigma(Xdx + Ydy + Zdz)$, and have seen that this equation holds good for a system restricted in any way, provided the *constraints are replaced by equivalent forces.*

If the constraints of the system consist of smooth curves or surfaces along which the points are restricted to move; of rigid bars or inextensible strings connecting the different points with each other or with any external fixed points; or in general of any connexions such that the distance between each pair of points immediately acting on each other is invariable, the whole work done by all these constraints and connexions is zero, and may therefore be omitted from the right-hand side of the equation (Arts. 124, 127).

If the potential energy (Arts. 129, 282) of *any portion* of the system be a function of a single variable quantity u, the work done by this part of the system in any displacement must be of the form $\lambda\delta u$; for $V = \phi(u)$, and therefore $dV = \phi'(u)\,du$.

If between any points of the system there be a connexion which is capable of being expressed by means of an equation

between their coordinates, such connexion can be effected by means of constraints of an invariable character; such as smooth fixed surfaces or curves, or rigid bars or inextensible strings.

Hence we may conclude that, *in any motion of the system, the work done by the forces replacing any connexion between the points of the system which is capable of being expressed by equations between their coordinates, is zero.*

A formal proof of this important proposition may be given as follows:—

If $U = 0$ be an equation between the coordinates of any points in a moving system, the forces which the corresponding constraint introduces into the system must be functions of the coordinates and of the other forces. Hence, if the latter be conservative, so are the forces caused by the constraint, which for brevity we shall refer to as the constraint U.

Again, if at any time the condition $U = 0$ be actually fulfilled, the imposition or removal of the material bonds by which the corresponding constraint is effected cannot require any expenditure of energy; since this imposition or removal does not change the position of any point of the system.

Let there be now a system S_1, which without U is conservative, and let A and B be two configurations in which the condition $U = 0$ is fulfilled; then, as we have seen, the forces replacing U are conservative, and if they consume work in the motion from B to A, they produce work in that from A to B. Let the external work W bring S_1 from A to B subject to the constraint U. Let Q be the amount of potential energy thereby produced in the system, and E the work done by the forces replacing U; then, the whole amount of work produced is $W - Q + E$. Now let this be used in bringing S_2 (precisely similar to S_1) from B to A without the constraint U, whereby Q is produced, and in doing such an amount of other work that S_1 may come to rest in the position B and S_2 in the position A. We may then without any expenditure of work impose the constraint U on S_2 and remove it from S_1. Things are now in precisely the same state as at starting, and in the whole process, by an expenditure of work W, we have produced work whose amount is $W + E$. Hence in any motion of the system the work E done by the forces replacing the condition $U = 0$ must be zero.

As the amount of work done by these forces in an unreversed motion cannot be influenced by the *character* of the other forces, but only by their amounts and directions, the work done by the forces replacing $U = 0$ must under any circumstances be zero.

285. **Equation of Energy in General.**—If we have a system acted on by any forces external or internal, and subject to any constraints or mutual connexions, the equation of energy assumes the form

$$T - T_0 + V - V_0 = W. \qquad (5)$$

T_0 and V_0 are the kinetic and potential energies in the initial position, T and V those in the position under consideration, and W the work done in going from the initial to the actual position by the external forces and by those internal forces which are not conservative or reversible in their character.

As regards constraints and connexions, they may be divided into three classes. 1. Those producing forces whose work during any motion of the system is zero. Such connexions we have already considered; they have no effect on the equation of energy. 2. Those which are capable of alteration under the action of external forces, and such that their alteration produces or consumes a corresponding amount of potential energy. The work done by the forces replacing these constraints and connexions is included in the expression $V_0 - V$. 3. Resistances or connexions which introduce forces of a non-conservative character. Such are the friction of rough surfaces, the resistance of a medium, the forces developed by the alteration of an extensible body which does less work in its recoil than the amount required to stretch it, &c. All such forces must appear as forces in the equation of energy, and the work done by them is included in W.

286. **Virtual Velocities.**—The principle of energy may, as we have seen, be expressed for dynamical purposes in the following form :—

The work done on a system in any interval of time by external applied forces, diminished by the work consumed in the same time by the non-conservative forces of the system, is equal to the sum of the increments of the kinetic and potential energies.

We have seen likewise that this principle holds good for a system subject to any invariable constraints or connexions internal or external as well as for a free system.

We are now able to obtain the conditions which must be fulfilled in order that any system should be in equilibrium; they can be expressed in a single statement, viz :—

In order that any system should be in equilibrium, the work done by the applied forces in any possible infinitely small displacement, diminished by the increase of the potential energy of the system, must be either negative or zero; and, if this be true for every possible infinitely small displacement, the system is in equilibrium.

The truth of this statement readily appears from the equation of energy.

A position of equilibrium is one in which if the system be placed at rest it will remain at rest. Now the system will not remain at rest if there be any possible mode of displacement, in which the united action of the internal and external forces can produce a velocity in any of the points of the system. On the other hand, if the system move from rest in any manner, it will acquire a positive kinetic energy. Hence, if there be no possible way in which it can do this, its position must be one of equilibrium.

In applying the principle of equilibrium we must regard the non-conservative forces of the system (if any) as applied forces, and introduce them with their proper signs. In the case of *actual* motion, forces of this kind always consume work, but in the case of *virtual* displacements this is not necessarily the case; *e. g.* suppose a heavy particle is placed on a rough inclined plane, and it is required to determine the condition of equilibrium. In this case we must consider the force of friction as acting upwards along the plane. If now we imagine a virtual displacement down the plane, friction will consume work; but if we imagine a displacement up the plane, friction will produce work. In the case of actual motion, whether slipping take place up or down the plane, friction will consume work.

Again it is to be observed, that if every possible set of displacements be also possible when reversed, the condition of

equilibrium becomes simply that *the total work done by all the forces internal and external be zero.*

In fact, if $\Sigma P\delta p$ be negative and P remaining unaltered the sign of each δp be changed, $\Sigma P\delta p$ becomes positive; but this is inconsistent with the principle of equilibrium as stated above; hence $\Sigma P\delta p$ must be zero.

If we combine the principle of Virtual Velocities with D'Alembert's principle, we obtain the equation which embraces the whole theory of Kinetics,

$$\Sigma\left\{\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\frac{d^2z}{dt^2}\right)\delta z\right\} = 0.$$

From this equation that of energy was deduced in Chapter IX. In the present chapter we have reversed this mode of procedure.

287. **Equivalent Sets of Forces.**—Two sets of forces acting on any material system are said to be equivalent when the motions produced by one set are identical with those produced by the other.

If each of two sets of forces be capable of equilibrating the same third set, the two are equivalent.

For let P be a force of the first set, Q one of the second, and R one of the set which each of the first two can equilibrate. Suppose the P set only to act. Introduce at the point where R would act two forces R and $-R$. This being done for each point of the system, the motion remains undisturbed. The system is now acted on by the three sets of forces P, R, and $-R$; and, since the sets P and R are in equilibrium, the sets P and $-R$ are equivalent. In like manner the sets Q and $-R$ are equivalent. Hence the sets P and Q are equivalent.

In moving a system from one given position to another, equivalent sets of forces produce the same amount of work.

The motion being the same whichever set of forces is in action, the intermediate positions of the system are at each instant the same; consequently, since the two sets of forces are each capable of equilibrating the same set, we have $\Sigma P\delta p = \Sigma Q\delta q$ at each instant. Hence the whole amount of work produced in one case is equal to that produced in the other.

It can be shown in like manner that the work required to move a system from one given position to another, against the

action of any set of forces, is equal to that required to move it against the action of an equivalent set.

288. **Wrenches.**—A wrench in Kinetics corresponds to a twist in Kinematics.

If a rigid body be acted on by any forces, these forces can be reduced to a single force along with a couple whose plane is perpendicular to the direction of the force.

Such a system is called a *wrench about a screw*, the axis of the screw being the line of direction of the force, and the pitch of the screw the line which is the quotient obtained by dividing the moment of the couple by the force. The magnitude of the force is called the intensity of the wrench.

The wrench to which a set of forces acting on a rigid body is equivalent has been termed the canonical form of the set of forces.

The canonical form of a set of forces is in general unique; for, as may be easily seen, if two wrenches be equivalent, they must either be identical or else consist of equal couples in parallel planes.

Examples.

1. A particle of mass m moves with a simple harmonic motion; determine its mean energy.

If τ and a be the periodic time and amplitude of the motion (Arts. 87, 88), and T the mean energy,

$$T = \frac{1}{\tau}\int_0^\tau \frac{mv^2}{2}\,dt = m\frac{\pi^2}{\tau^2}a^2.$$

2. If the motion of the particle m be the resultant of any number of simple harmonic motions having different periods and amplitudes, find the mean value of the energy.

If θ be an interval of time which is very great compared with the longest periodic time,

$$T = \frac{1}{\theta}\int_0^\theta \frac{mv^2}{2}\,dt = m\pi^2\Sigma\frac{a^2}{\tau^2}.$$

3. Determine the mean energy of a system of vibrating particles.

The rectangular components of the displacement of any particle are periodic functions of the time, and can therefore be expanded in a series of terms of the form

$$a\sin\left(\frac{2\pi}{\tau}t + \alpha\right).$$

Hence,
$$T = \pi^2\Sigma m\,\frac{a^2 + b^2 + c^2}{\tau^2}.$$

4. A rigid body is acted on by a couple whose moment is Pp; determine the work done by the couple in any small motion of the body.

If $d\theta$ be the angular displacement of the body round an axis perpendicular to the plane of the couple, the work done by the couple is $Pp d\theta$, *see* Art. 128.

5. Express the kinetic energy of a body having a fixed point in terms of the angles θ, ϕ, ψ (Art. 258), the body-axes being the principal axes at the fixed point.

As ω_1, ω_2, ω_3 are given in terms of θ, ϕ, ψ, Ex. 5, Art. 260, we have, Art. 263,

$$2T = (A\sin^2\phi + B\cos^2\phi)\,\dot\theta^2 + \{(A\cos^2\phi + B\sin^2\phi)\sin^2\theta + C\cos^2\theta\}\,\dot\psi^2$$
$$+ C\dot\phi^2 + 2(B-A)\,\dot\theta\dot\psi\sin\theta\sin\phi\cos\phi + 2C\dot\phi\dot\psi\cos\theta.$$

6. If T be the kinetic energy of a body having a fixed point, and ω_x, ω_y, ω_z its angular velocities round three rectangular axes fixed in space passing through the point, prove that $\frac{dT}{d\omega_x}$, $\frac{dT}{d\omega_y}$, $\frac{dT}{d\omega_z}$ are the moments of momentum of the body round the axes.

Let $\dot x$, $\dot y$, $\dot z$ be the components of the velocity of any point of the body, then

$$2T = \Sigma m(\dot x^2 + \dot y^2 + \dot z^2), \quad \frac{dT}{d\omega_x} = \Sigma m\left(\dot x\frac{d\dot x}{d\omega_x} + \dot y\frac{d\dot y}{d\omega_x} + \dot z\frac{d\dot z}{d\omega_x}\right);$$

hence (Art. 255),

$$\frac{dT}{d\omega_x} = \Sigma m(y\dot z - z\dot y).$$

7. Determine the amount of energy W which must be expended on a rigid body in order to effect a given twist in opposition to a given wrench.

Let θ be the amplitude of the twist round the screw α whose pitch is p, Q the intensity of the wrench round the screw β whose pitch is q, and Ω the angle between α and β.

Take α as axis of x, and the shortest distance from α to β as axis of z, denoting its length by c. Replace the wrench at each instant by the forces X, Y, Z, passing through a point coinciding with the origin, and the couples L, M, N. Then the system X, Y, Z, L, M, N being equivalent to the wrench,

$$X = Q\cos\Omega, \quad Y = Q\sin\Omega, \quad Z = 0,$$

$$L = Qq\cos\Omega - Qc\sin\Omega, \quad M = Qq\sin\Omega + Qc\cos\Omega, \quad N = 0.$$

Hence (Ex. 4, and Art. 123),

$$dW = Qp\cos\Omega d\theta + (Qq\cos\Omega - Qc\sin\Omega)\,d\theta = Q\{(p+q)\cos\Omega - c\sin\Omega\}\,d\theta,$$

and therefore

$$W = Q\{(p+q)\cos\Omega - c\sin\Omega\}\,\theta.$$

The expression $(p+q)\cos\Omega - c\sin\Omega$ is called (Ball, *Theory of Screws*, § 13) the virtual coefficient of the pair of screws α and β.

If c be regarded as always positive, Ω is the angle through which the axis of the screw α must be turned round the axis of z in order to be codirectional with the axis of β, the positive direction of z being always from α towards β.

8. A smooth rod having one extremity fixed moves on a smooth horizontal table, and drives a particle of mass equal to its own, which starts from rest from a point indefinitely near the fixed extremity of the rod. Find the inclination of the rod to the direction of motion of the particle when the latter has reached any definite point of the rod.

As the moment of momentum and the *vis viva* of the system are constant (*see* Art. 201), we have, if ψ be the angle required, $\tan\psi = \frac{k}{\sqrt{(r^2+k^2)}}$, where k is the radius of gyration of the rod and r the distance of the particle from the fixed extremity.

9. A triangular prism rests with one rectangular face on a smooth horizontal plane. A rough cylinder, having its axis parallel to the edge of the prism, rolls down one of its faces starting from rest, the centres of inertia of the prism and cylinder being in the same vertical plane; determine the angular velocity of the cylinder when it reaches the horizontal plane, and the distance through which the prism has moved.

Let the axis of x be the intersection of the horizontal plane with the vertical plane containing the centres of inertia, the axis of z being vertical. Let x' be the coordinate of the centre of inertia of the prism, m' its mass; x, z the coordinates of the centre of the cylinder, a its radius, m its mass, k its radius of gyration, ϕ the angle through which it has turned, and s the distance on the prism perpendicular to its edge through which the line of contact of the cylinder has moved, at any time; then, i being the angle which the face of the prism makes with the horizontal plane,

$$x - x' = x_0 - x_0' + s\cos i, \quad z = z_0 - s\sin i, \quad s = a\phi;$$

and from the conservation of the motion of the centre of inertia and the equation of *vis viva*, we have

$$mx + m'x' = mx_0 + m'x_0', \quad m'\dot{x}'^2 + m(\dot{x}^2 + \dot{z}^2 + k^2\dot{\phi}^2) = 2gm(z_0 - z).$$

Hence, if h be the initial height of the centre of the cylinder, and ω its angular velocity when it reaches the horizontal plane,

$$\omega^2 = \frac{2(m+m')g(h-a)}{(m' + m\sin^2 i)a^2 + (m+m')k^2}, \quad x' - x_0' = -\frac{m}{m+m'}(h-a)\cot i.$$

10. Show that the velocity v with which a fluid, under a uniform pressure p, escapes from a small orifice is given by the equation $v^2 = 2gh$, where h is the height of a column of the fluid which would produce the pressure p.

Suppose a small mass m of fluid forced through an orifice, whose section is σ, into a large volume of fluid under the pressure p. If x be the distance through which the surface σ of the fluid is pushed in by this operation, the work expended is $px\sigma$.

Hence the potential energy lost when m escapes is $px\sigma$, and this must be the kinetic energy acquired by m, therefore $\dfrac{mv^2}{2} = px\sigma$. Now, if ρ be the density of the fluid, $p = g\rho h$, and $m = \rho x\sigma$. Substituting, we have

$$v^2 = 2gh.$$

11. Determine the total energy, kinetic and potential, of a planet and satellite moving as in Ex. 3, Art. 213.

If Y be the total energy

$$2Y = C\left\{n^2 - \frac{\mu Mm}{C} r^{-1}\right\} + K,$$

where K is an undetermined constant, and C, &c., have the same significations as in the example referred to.

12. A planet and a satellite move as in the last example. If with a given moment of momentum it is possible to set them moving as a rigid body, it is possible to do so in two ways—for one of which the energy is a maximum, and for the other a minimum.

If in the equation of Ex. 3, Art. 213, we substitute x for $r^{\frac{1}{2}}$, y for n, and put h for the moment of momentum, we have

$$h = Cy + \mu^{\frac{1}{2}} Mm(M + m)^{-\frac{1}{2}} x.$$

Again (Ex. 11),

$$2Y - K = Cy^2 - \mu Mm\frac{1}{x^2}.$$

By a proper selection of the units of mass, length and time, we can make C, $\mu^{\frac{1}{2}}Mm(M+m)^{-\frac{1}{2}}$, and μMm each equal to unity. We obtain thus for the unit of mass $\dfrac{Mm}{M+m}$, for that of length $\left\{\dfrac{C(M+m)}{Mm}\right\}^{\frac{1}{2}}$, and for that of time $\left\{\dfrac{C^3(M+m)}{\mu^2 M^3 m^3}\right\}^{\frac{1}{2}}$. We have then

$$h = y + x, \quad 2Y - K = y^2 - \frac{1}{x^2}.$$

If the whole system move as a rigid body, the angular velocity ω of the satellite round the centre of inertia of the system must be equal to n; but $r^3 = \mu(M + m)\omega^{-2}$, and in the special units adopted $\mu^{\frac{1}{2}}(M + m)^{\frac{1}{2}} = 1$; hence $x^3 = \dfrac{1}{y}$. Again, if Y be maximum or minimum, we have

$$\frac{d}{dx}\left\{(h - x)^2 - \frac{1}{x^2}\right\} = 0, \quad \text{or} \quad x - h + \frac{1}{x^3} = 0,$$

which is the same equation for determining x as before. Hence, *if the whole system move as a rigid body*, the total energy Y is a maximum or a minimum.

Again, let $f(x) = x^4 - hx^3 + 1 = x^3(x - h) + 1$. If x be negative $f(x)$ is positive, and therefore the biquadratic $f(x) = 0$ has no negative root, and cannot therefore have more than two real roots, since the coefficient of x^2 is zero. If $x > h$, $f(x)$ is positive, and therefore the biquadratic has no positive root greater than h;

but if x be positive and less than h, $f(x)$ may be negative, and therefore the biquadratic may have two positive roots between 0 and h. As $f'(x)=x^2(4x-3h)$, if the biquadratic have two real roots, one is greater than $\frac{3}{4}h$ and the other less. The greater root makes $f'(x)$, and therefore $\frac{d^2Y}{dx^2}$ positive, and Y a minimum; the lesser root makes $\frac{d^2Y}{dx^2}$ negative, and Y a maximum.

13. Apply the preceding examples to determine the secular effects of tidal friction on the Earth-moon system, the moon being supposed to move in the plane of the equator.

If the Earth's radius be denoted by a, C is approximately $\frac{1}{3}Ma^2$, and $M=82m$. Hence the unit of mass is $\frac{M}{83}$, the unit of length $5\cdot26a$, and the unit of time 2 hours 41 minutes. Again, in the special units, the present value of r is $11\cdot454$, and that of n is $0\cdot704$, whence x is $3\cdot384$; also $h=4\cdot088$. It is plain that for this value of h the biquadratic $f(x)=0$ has two real roots. The lesser of these, x_1, makes Y a maximum, and the greater, x_2, makes Y a minimum. Again, $f(x)$ is positive for values of x between 0 and x_1, negative for those between x_1 and x_2, and positive for those greater than x_2. As x is positive throughout, when $f(x)$ is positive we have $\frac{1}{x^3}>y$, i.e. $\omega>n$; and $\frac{1}{x^3}<y$, i.e. $\omega<n$, when $f(x)$ is negative. At present $f(x)$ is negative, and therefore the present state of things corresponds to a value of x which lies between x_1 and x_2.

We can now determine the effects of tidal friction. Since the friction resulting from the lunar tides constantly diminishes the sensible or mechanical energy of the Earth-moon system, Y must continually decrease (Art. 282). Hence, as at present $f(x)$ is negative, x must increase and y decrease until Y reaches its minimum, after which the whole system will move as if rigidly connected.

It appears accordingly that the friction caused by the lunar tides diminishes the angular velocity of the Earth, *i.e.* increases the length of the day, and at the same time increases the Moon's distance and the length of the month. This process must go on till the day and month are of equal length, after which the lunar tides will cease. If at any past period the Moon moved as if rigidly connected with the Earth, this must have been when Y was a maximum. Such a state of things was dynamically unstable, for the least disturbance of the rigidity of the motion would have produced tides whose friction would have diminished the energy, and caused the system to depart farther from the configuration of maximum energy. The departure from this configuration might have taken place in two ways, according as the Moon approached the Earth or receded from it. If the former event had occurred, the Moon's angular velocity in its orbit would have become greater than the angular velocity of the Earth's rotation, and the Moon must ultimately have fallen upon the Earth, as x must have decreased continually along with Y. If on the other hand the Moon had receded, the present state of things would have been reached. The value of x which makes Y a minimum lies between $4\cdot073$ and $4\cdot074$, and the corresponding value of n lies between $0\cdot015$ and $0\cdot014$. The ratios of the present value of n to these two values are $46\cdot9$ and $50\cdot2$. The present investigation would therefore lead us to conclude that, when the lunar tides cease and the day and month become equal, the length of the day will be between 46 and 50 times its present length.

Examples 11, 12, 13, and Example 3, Art. 213, are taken from a Paper by Professor G. H. Darwin, first published in the *Proceedings* of the Royal Society for 1879, and subsequently, with some alterations, in Thomson and Tait's *Natural Philosophy*, Part ii. In this Paper Mr. Darwin gives diagrams of the curves represented by the equations

$$\eta = 2Y - K = F(x), \quad x^3 y = 1, \quad h = x + y,$$

by means of which the results arrived at are exhibited to the eye.

14. A great number of smooth perfectly elastic particles are moving with great velocity in various directions within a rectangular parallelepiped, two of whose opposite faces are large compared with the others. If one of these faces be movable, determine the force required to keep it steady.

Let u be the velocity of one of the particles whose mass is m, and ϕ the angle which the direction of its motion makes with the normal to the face. Before striking the face the particle has a normal velocity $u \cos \phi$, and after the shock it acquires an equal normal velocity in the opposite direction. The momentum communicated to the face is therefore $2mu \cos \phi$. Having reached the opposite face, the particle rebounds and strikes the movable face again; the interval of time between two successive shocks against the movable face being $\dfrac{2a}{u \cos \phi}$, where a is the perpendicular distance between the faces. The whole momentum communicated to the movable force by the particle m in the time θ is therefore $\dfrac{mu^2 \cos^2\phi}{a}\,\theta$, and the whole momentum M communicated by all the particles in the same time is $\dfrac{\theta}{a}\Sigma mu^2 \cos^2\phi$.

In order to determine the value of $\Sigma mu^2 \cos^2\phi$, describe a sphere of unit radius, and draw from its centre lines parallel to the directions of motion of the various particles at the beginning of the interval of time θ. Since the number of particles is very great and the direction of the motion of any one undetermined, we may assume that the energy of those particles whose directions of motion make an angle ϕ with a fixed direction is to the total energy of the system as that portion of the surface of the sphere comprised between the cones whose semi-angles are ϕ and $\phi + d\phi$ is to the whole surface. If T be the total energy of the moving particles, we have then

$$\Sigma mu^2 \cos^2\phi = T\int_0^\pi \cos^2\phi \sin\phi\, d\phi = \frac{2}{3}T.$$

Hence $M = \dfrac{2}{3}\dfrac{T\theta}{a}$. Now the required force F must be such as to communicate to the movable face the momentum M in the time θ, and therefore

$$F\theta = M = \frac{2}{3}\frac{T\theta}{a}, \text{ or } F = \frac{2}{3}\frac{T}{a}.$$

15. A number of particles move as in the last example; determine the pressure which they exert on the unit of area.

If S be the area of the movable face in the last example, and p the pressure of the particles on the unit of area, $pS = F = \frac{2}{3}\frac{T}{a}$. Hence, if v be the volume of the parallelepiped, $pv = \frac{2}{3}T$.

The results obtained in Ex. 14 and 15 are made use of to explain the pressure which a gas exerts against its envelope. The mode of investigation employed is due to Clausius.

16. Determine the mean kinetic energy of any system in stationary motion.

A system is said to be in stationary motion when the coordinates and the velocities of its various points fluctuate within determinate finite limits.

If we integrate $\dot{x}^2 dt$ by parts, we get $\int \dot{x}^2 dt = x\dot{x} - \int x\ddot{x}\,dt$; and similar equations may be obtained corresponding to the other coordinates. Again, supposing each point of the system to be free, we have $m\ddot{x} = X$. Hence, if $\theta = t_1 - t_0$,

$$\frac{1}{\theta}\int_{t_0}^{t_1} T dt = \frac{1}{2\theta}\Sigma m\{x_1\dot{x}_1 + y_1\dot{y}_1 + z_1\dot{z}_1 - (x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0)\}$$

$$- \frac{1}{2\theta}\Sigma\int_{t_0}^{t_1}(Xx + Yy + Zz)\,dt.$$

If θ be made sufficiently large, the first term on the right-hand side of this equation may be neglected, and we find that the mean value of T is equal to the mean value of

$$-\tfrac{1}{2}\Sigma(Xx + Yy + Zz).$$

This latter quantity is termed by Clausius the *virial* of the system. Hence, *the mean kinetic energy is equal to the virial.* This theorem, and its application given in Ex. 17, 18, are due to Clausius, whose investigation will be found in the *Philosophical Magazine* for August, 1870.

17. If Π be the virial of a system which is acted on by no external forces except a uniform pressure on its surface, prove that

$$\Pi = \tfrac{3}{2}pv - \frac{1}{\theta}\int_{t_0}^{t_1}\Sigma r\phi(r)\,dt,$$

where r is the distance between any two points of the system, v its volume, and p the pressure which it exerts upon the unit area of the surface of the surrounding medium or envelope.

If r be the distance between two particles of the system whose coordinates are x, y, z; x', y', z', the portion of Π due to the mutual action of these particles is the mean value of an expression of the form

$$-\phi(r)\left(\frac{x-x'}{r}x + \frac{y-y'}{r}y + \frac{z-z'}{r}z\right) - \phi(r)\left(\frac{x'-x}{r}x' + \frac{y'-y}{r}y' + \frac{z'-z}{r}z'\right),$$

or $\;-r\phi(r)$. (Art. 283.)

Again, if dS be an element of the bounding surface of the system the direction cosines of whose normal are l, m, n, the part of Π due to the external pressure is

$$\tfrac{1}{2}p\iint(xl+ym+zn)\,dS = \tfrac{1}{2}p\left\{\iint x\,dy\,dz + \iint y\,dz\,dx + \iint z\,dx\,dy\right\} = \tfrac{3}{2}pv.$$

Hence $$\Pi = \tfrac{3}{2}pv - \frac{1}{\theta}\int_{t_0}^{t_1}\Sigma r\phi(r)\,dt.$$

18. Determine the pressure of a gas in terms of its volume and the mean kinetic energy of translation of its molecules.

A gas may be regarded as composed of a number of molecules which exert no action on each other except when in contact. If the gas be enclosed in an envelope, and its condition remain unaltered, its molecules must be in stationary motion. Hence, if T' be the mean value of that part of the kinetic energy which results from the velocities of the centres of inertia of the molecules, and Π the corresponding virial, we have $T' = \Pi$; but $\Pi = \tfrac{3}{2}pv$ (Ex. 17), since the time during which a particle is in contact with other particles is negligible compared with the interval between two such contacts, and therefore the other term of Π may in this case be neglected. Accordingly $pv = \tfrac{2}{3}T'$.

Section II.—*The General Equations of Dynamics.*

289. **General Equations of Motion for any System.**—The general equations of motion for any system are obtained in precisely the same manner as the general equations of equilibrium.

If $F = 0$, $G = 0$, $H = 0$, &c., are the equations of condition representing the connexions and constraints, we have

$$\frac{dF}{dx_1}\delta x_1 + \frac{dF}{dy_1}\delta y_1 + \frac{dF}{dz_1}\delta z_1 + \frac{dF}{dx_2}\delta x_2 + \&\text{c.} = 0.$$

$$\frac{dG}{dx_1}\delta x_1 + \&\text{c.} = 0,\quad \frac{dH}{dx_1}\delta x_1 + \&\text{c.} = 0,\ \&\text{c.}$$

Multiply the first by λ, the second by μ, the third by ν, &c., and add to D'Alembert's Equation, and we obtain

$$\left(X_1 - m_1\frac{d^2x_1}{dt^2} + \lambda\frac{dF}{dx_1} + \mu\frac{dG}{dx_1} + \nu\frac{dH}{dx_1} + \&\text{c.}\right)\delta x_1 + \&\text{c.} = 0. \quad (1)$$

If there be n equations of condition we can assign to λ, μ, ν, &c., such values as to make the coefficients of the first n displacements in the above equation vanish. By means of

these displacements we can satisfy the n equations $\delta F = 0$, $\delta G = 0$, &c. The remaining displacements are then entirely unrestricted, and their coefficients in (1) must therefore be each zero, and we have for the equations of motion

$$\left.\begin{aligned} m_1 \frac{d^2x_1}{dt^2} &= X_1 + \lambda \frac{dF}{dx_1} + \mu \frac{dG}{dx_1} + \nu \frac{dH}{dx_1} + \text{\&c.} \\ m_1 \frac{d^2y_1}{dt^2} &= Y_1 + \lambda \frac{dF}{dy_1} + \mu \frac{dG}{dy_1} + \nu \frac{dH}{dy_1} + \text{\&c.} \\ m_1 \frac{d^2z_1}{dt^2} &= Z_1 + \lambda \frac{dF}{dz_1} + \mu \frac{dG}{dz_1} + \nu \frac{dH}{dz_1} + \text{\&c.} \\ m_2 \frac{d^2x_2}{dt^2} &= X_2 + \lambda \frac{dF}{dx_2} + \mu \frac{dG}{dx_2} + \nu \frac{dH}{dx_2} + \text{\&c.} \\ \text{\&c.,} \qquad & \qquad \text{\&c.,} \qquad\qquad \text{\&c.} \end{aligned}\right\} \qquad (2)$$

From these equations we can obtain the Equation of Energy, if we multiply the first by dx_1, the second by dy_1, &c., and add. In this manner we obtain

$$dT = \Sigma\,(Xdx + Ydy + Zdz) + \lambda\left(\frac{dF}{dx_1}dx_1 + \frac{dF}{dy_1}dy_1 + \text{\&c.}\right) + \text{\&c.}$$

Now, if the equation $F = 0$ involve only the coordinates of the various points,

$$\frac{dF}{dx_1}dx_1 + \frac{dF}{dy_1}dy_1 + \text{\&c.} = dF = 0,$$

and the condition expressed by the equation $F = 0$ has no effect on the kinetic energy.

This result was obtained from first principles in Art. 284, and by its means the Equation of Virtual Velocities in its usual form was deduced from the Equation of Energy.

290. **Equation of Energy when Equations of Condition Involve the Time Explicitly.**—If the equation

$F = 0$ involve the time explicitly, the work done in any actual motion of the system by the forces capable of replacing the condition $F = 0$ need not be zero. In a virtual displacement the work done by these forces must still be zero, because in such a displacement no lapse of time is supposed to take place. So far, therefore, as the equation of virtual velocities is concerned, t must be considered constant in the equation $F = 0$, and as in Art. 200 the virtual displacements must fulfil the condition

$$\frac{dF}{dx_1}\delta x_1 + \frac{dF}{dy_1}\delta y_1 + \frac{dF}{dz_1}\delta z_1 + \frac{dF}{dx_2}\delta x_2 + \&c. = 0.$$

The actual displacements on the other hand fulfil the condition

$$\frac{dF}{dx_1}dx_1 + \frac{dF}{dy_1}dy_1 + \frac{dF}{dz_1}dz_1 + \frac{dF}{dx_2}dx_2 + \&c. + \left(\frac{dF}{dt}\right)dt = 0.$$

Hence in this case the Equation of Energy becomes

$$dT = \Sigma\,(Xdx + Ydy + Zdz) - \lambda\left(\frac{dF}{dt}\right)dt - \mu\left(\frac{dG}{dt}\right)dt - \&c. \quad (3)$$

291. **Similar Mechanical Systems.**—Two systems are geometrically similar when each line of the one is equal to the corresponding line of the other multiplied by the same constant.

Similar Mechanical systems are not only geometrically similar, but have also a similar distribution of mass, and a similar distribution of force, and work in a similar manner; *i.e.* each mass of the one is equal to the corresponding mass of the other multiplied by a constant, each force of the one is equal to the corresponding force of the other multiplied by a constant; and the systems are always geometrically similar at instants of time whose intervals from two fixed epochs are in a constant ratio.

Let x be a coordinate of a point in the first system, m a mass, X a force, and t an interval of time; and x', m', X', t'

the corresponding quantities for the second system; we have then the equations $x' = lx$, $m' = \mu m$, $X' = \lambda X$, $t' = \nu t$.

Hence, $$\Sigma m'\left(\frac{d^2x'}{dt'^2}\delta x' + \frac{d^2y'}{dt'^2}\delta y' + \frac{d^2z'}{dt'^2}\delta z'\right)$$
$$= \frac{\mu l^2}{\nu^2}\Sigma m\left(\frac{d^2x}{dt^2}\delta x + \frac{d^2y}{dt^2}\delta y + \frac{d^2z}{dt^2}\delta z\right),$$

and $$\Sigma(X'\delta x' + Y'\delta y' + Z'\delta z') = \lambda l \Sigma(X\delta x + Y\delta y + Z\delta z).$$

In order, therefore, that D'Alembert's equation should hold good for each system, we must have $\mu l = \lambda\nu^2$.

This equation of condition may be put into another form by expressing ν in terms of the ratio of the corresponding velocities in the two systems. If we denote this ratio by a, we have $\frac{dx'}{dt'} = a\frac{dx}{dt}$, but also, $\frac{dx'}{dt'} = \frac{l}{\nu}\frac{dx}{dt}$; therefore $l = a\nu$, and the equation of condition becomes $\lambda l = \mu a^2$.

If, as is generally the case, gravity be one of the moving forces in both systems, we must have $\lambda = \mu$; hence $a^2 = l$, or the velocity in each system must be proportional to the square root of its linear dimensions.

292. **Generalized Coordinates.**—If a system have n degrees of freedom its position is completely determined at each instant by the values of n independent variables, which may be termed *coordinates*, and be denoted by $\xi_1, \xi_2, \xi_3, \ldots \xi_n$. The Cartesian coordinates x, y, z of any point of the system are expressible in terms of these new coordinates, and are therefore functions of the n variables ξ_1, ξ_2, &c., these latter being functions of the time.

If we differentiate the equation $x = f(\xi_1, \xi_2, \xi_3, \ldots \xi_n)$ with respect to the time we obtain

$$\dot{x} = \frac{dx}{d\xi_1}\dot{\xi}_1 + \frac{dx}{d\xi_2}\dot{\xi}_2 \ldots + \frac{dx}{d\xi_n}\dot{\xi}_n. \quad (4)$$

This equation shows that $\dot{x}$ is a function of the velocities $\dot{\xi}_1$,

&c., and of the coordinates ξ_1, &c., and is linear with respect to the velocities. From (4) we have

$$\frac{d\dot{x}}{d\dot{\xi}_1} = \frac{dx}{d\xi_1}, \quad \frac{d\dot{x}}{d\dot{\xi}_2} = \frac{dx}{d\xi_2}, \text{ \&c.} \tag{5}$$

Again, if we differentiate $\frac{dx}{d\xi_1}$ with respect to t, we get

$$\frac{d}{dt}\frac{dx}{d\xi_1} = \frac{d^2x}{d\xi_1^2}\dot{\xi}_1 + \frac{d^2x}{d\xi_1\,d\xi_2}\dot{\xi}_2 \ldots + \frac{d^2x}{d\xi_1\,d\xi_n}\dot{\xi}_n;$$

but by (4) this is the expression for the partial differential coefficient $\frac{d\dot{x}}{d\xi_1}$. Hence we have

$$\frac{d}{dt}\frac{dx}{d\xi_1} = \frac{d\dot{x}}{d\xi_1}, \quad \frac{d}{dt}\frac{dx}{d\xi_2} = \frac{d\dot{x}}{d\xi_2}, \text{ \&c.} \tag{6}$$

Any set of n independent variables which completely determine the position of a system may be taken as the generalized coordinates of the system. The number of these coordinates is fixed, but the actual coordinates are in general to a great extent arbitrary.

293. **Kinetic Energy and Generalized Coordinates.**—The kinetic energy T of any system in motion is given by the equation $2T = \Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ If we substitute for $\dot{x}$, $\dot{y}$, &c., their values given by (4) and the corresponding equations, we obtain a homogeneous quadratic function of the n velocities $\dot{\xi}_1$, $\dot{\xi}_2$, . . . $\dot{\xi}_n$, the coefficients of $\dot{\xi}_1^2$, $\dot{\xi}_1\dot{\xi}_2$, &c., being functions of the coordinates ξ_1, ξ_2, &c., and of the constants of the system. If we denote these coefficients by $\mathfrak{X}_{11}$, $2\mathfrak{X}_{12}$, &c., we have the equation

$$2T = \mathfrak{X}_{11}\dot{\xi}_1^2 + \mathfrak{X}_{22}\dot{\xi}_2^2 + \text{\&c.} + 2\mathfrak{X}_{12}\dot{\xi}_1\dot{\xi}_2 + 2\mathfrak{X}_{13}\dot{\xi}_1\dot{\xi}_3 + \text{\&c.} \tag{7}$$

294. **Equations of Motion for Impulses.**—If a system start from rest under the action of any set of impulses X, Y, Z, &c., the initial velocities are determined from D'Alembert's equation by equating to zero the coefficient of

each independent variation. Now, if ξ_1, ξ_2, &c. be the generalized coordinates,

$$\delta x = \frac{dx}{d\xi_1}\delta\xi_1 + \frac{dx}{d\xi_2}\delta\xi_2 + \&c., \ \delta y = \frac{dy}{d\xi_1}\delta\xi_1 + \frac{dy}{d\xi_2}\delta\xi_2 + \&c.,$$

where $\delta\xi_1$, $\delta\xi_2$, &c. are independent arbitrary variations. Hence, substituting for δx, δy, &c. in D'Alembert's equation, we obtain as the equation of motion corresponding to the variation $\delta\xi_1$,

$$\Sigma m\left(\dot{x}\frac{dx}{d\xi_1} + \dot{y}\frac{dy}{d\xi_1} + \dot{z}\frac{dz}{d\xi_1}\right) = \Sigma\left(\underset{\cdot}{X}\frac{dx}{d\xi_1} + \underset{\cdot}{Y}\frac{dy}{d\xi_1} + \underset{\cdot}{Z}\frac{dz}{d\xi_1}\right).$$

The left-hand member of this equation becomes, if we sub-sttiute for $\dfrac{dx}{d\xi_1}$, $\dfrac{dy}{d\xi_1}$, &c. their values given by (5),

$$\Sigma m\left(\dot{x}\frac{d\dot{x}}{d\dot{\xi}_1} + \dot{y}\frac{d\dot{y}}{d\dot{\xi}_1} + \dot{z}\frac{d\dot{z}}{d\dot{\xi}_1}\right) \text{ or } \frac{dT}{d\dot{\xi}_1}.$$

Hence, if we put

$$\Sigma\left(\underset{\cdot}{X}\frac{dx}{d\xi_1} + \underset{\cdot}{Y}\frac{dy}{d\xi_1} + \underset{\cdot}{Z}\frac{dz}{d\xi_1}\right) = \underset{\cdot}{\Xi}_1, \ \&c.,$$

we have for a system starting from rest

$$\frac{dT}{d\dot{\xi}_1} = \underset{\cdot}{\Xi}_1, \ \frac{dT}{d\dot{\xi}_2} = \underset{\cdot}{\Xi}_2, \ \ldots \ \frac{dT}{d\dot{\xi}_n} = \underset{\cdot}{\Xi}_n. \tag{8}$$

In these equations T is supposed to be given by (7), and $\underset{\cdot}{\Xi}_1$, &c., are the generalized resultant components of the impulses tending to alter $\dot{\xi}_1$, &c.

It follows from what precedes that, for a system starting from rest, D'Alembert's equation

$$\Sigma m(\dot{x}\delta x + \dot{y}\delta y + \dot{z}\delta z) = \Sigma(\underset{\cdot}{X}\delta x + \underset{\cdot}{Y}\delta y + \underset{\cdot}{Z}\delta z)$$

becomes by transformation of coordinates

$$\frac{dT}{d\dot{\xi}_1}\delta\xi_1 + \frac{dT}{d\dot{\xi}_2}\delta\xi_2 + \&c. = \underset{\cdot}{\Xi}_1\delta\xi_1 + \underset{\cdot}{\Xi}_2\delta\xi_2 + \&c.$$

This equation may be written in the form

$$\Sigma\left(\frac{dT}{d\dot{\xi}}\,\delta\xi\right) = \Sigma\,(\Xi\,\delta\xi). \tag{9}$$

If a system be in motion, and p_1, p_2, &c. be the generalized impulse components which would give its actual motion to the system starting from rest, these impulses p_1, &c. are determined by the equations

$$p_1 = \frac{dT}{d\dot{\xi}_1}, \quad p_2 = \frac{dT}{d\dot{\xi}_2}, \; \ldots \; p_n = \frac{dT}{d\dot{\xi}}. \tag{10}$$

The differential coefficients $\frac{dT}{d\dot{\xi}_1}$, $\frac{dT}{d\dot{\xi}_2}$, &c. are the generalized components of momentum of the moving system.

If a system in motion be acted on by any set of impulses whose generalized components are Ξ_1, Ξ_2, &c., the changes of velocity produced by these impulses are given by the equations

$$\frac{dT}{d\dot{\xi}_1} - \left(\frac{dT}{d\dot{\xi}_1}\right)' = \Xi_1, \quad \frac{dT}{d\dot{\xi}_2} - \left(\frac{dT}{d\dot{\xi}_2}\right)' = \Xi_2, \text{ \&c.}, \tag{11}$$

where $\left(\frac{dT}{d\dot{\xi}_1}\right)'$, &c. correspond to the instant immediately before, and $\frac{dT}{d\dot{\xi}_1}$, &c. to that immediately after the action of the impulses. Since the values of ξ_1, &c. remain unaltered during the impulse, equations (11) may by (7) be written

$$\left.\begin{array}{l} \mathfrak{X}_{11}(\dot{\xi}_1 - \dot{\xi}_1') + \mathfrak{X}_{12}(\dot{\xi}_2 - \dot{\xi}_2') \ldots + \mathfrak{X}_{1n}(\dot{\xi}_n - \dot{\xi}_n') = \Xi_1 \\ \mathfrak{X}_{12}(\dot{\xi}_1 - \dot{\xi}_1') + \mathfrak{X}_{22}(\dot{\xi}_2 - \dot{\xi}_2') \ldots + \mathfrak{X}_{2n}(\dot{\xi}_n - \dot{\xi}_n') = \Xi_2 \\ \quad \text{\&c.} \qquad\qquad\qquad\qquad \text{\&c.} \qquad \text{\&c.} \end{array}\right\}. \tag{12}$$

295. **Kinetic Energy and Components of Momentum.**—Since T is a homogeneous quadratic function of $\dot{\xi}_1$, $\dot{\xi}_2$, &c., we have, by Euler's Theorem,

$$2T = \dot{\xi}_1 \frac{dT}{d\dot{\xi}_1} + \dot{\xi}_2 \frac{dT}{d\dot{\xi}_2} + \&c. = p_1 \dot{\xi}_1 + p_2 \dot{\xi}_2 \ldots . p_n \dot{\xi}_n. \quad (13)$$

This equation may be written in the abbreviated form

$$2T = \Sigma (p\dot{\xi}). \quad (14)$$

If we suppose the same system occupying the same position to have successively two different motions, in the first of which the velocity-components are $\dot{\xi}_1$, &c., and in the second, $\dot{\xi}_1'$, &c., and if we put $\dot{\xi}_1' = \dot{\xi}_1 + a_1$, &c., and express the corresponding values of T by $T_{\dot{\xi}}$ and $T_{\dot{\xi}'}$, we have, by Taylor's Theorem,

$$T_{\dot{\xi}'} = T_{\dot{\xi}} + \Sigma p a + T_a, \text{ i.e. } T_{\dot{\xi}'} = T_{\dot{\xi}} + \Sigma p (\dot{\xi}' - \dot{\xi}) + T_{(\dot{\xi}' - \dot{\xi})}. \quad (15)$$

If now we suppose a system to start from rest the values of certain components of velocity being prescribed, and if the system be set in motion by impulses such that there are no components of impulse except those corresponding to the prescribed velocities, the initial kinetic energy is a minimum.

Let $\dot{\xi}_1$, &c., be the velocity-components of the initial motion produced in the manner described, then p_1, &c., are the impulse-components; and if any impulse-component p_q be not zero, the corresponding velocity-component $\dot{\xi}_q$ is prescribed. Let us now suppose the system to be set in motion in any other way, the prescribed velocity-components being the same as before, and let $\dot{\xi}_1'$, &c., be the velocity-components of the new initial motion. We have, then, $\Sigma p (\dot{\xi}' - \dot{\xi}) = 0$, since whenever p is not zero, $\dot{\xi}' = \dot{\xi}$. Hence $T_{\dot{\xi}'} = T_{\dot{\xi}} + T_{(\dot{\xi}' - \dot{\xi})}$, and therefore $T_{\dot{\xi}}$ is a minimum.

This is Thomson's Theorem, Art. 199.

Again, if $\dot{\xi}_1$, &c., p_1, &c. be the components of velocity and momentum of a system in any given position, and $\dot{\xi}_1'$, &c., p_1', &c. the corresponding quantities for a different motion of the same system in the same position, we have

$$\Sigma (p\dot{\xi}') = \Sigma (p'\dot{\xi}). \quad (16)$$

The truth of this equation appears readily by substituting $\dot{\xi}_1 + \dot{\xi}_1'$, &c. for $\dot{\xi}_1$, &c. in $T_{\dot{\xi}}$, and equating the two expressions which by Taylor's Theorem can thence be obtained.

296. **Energy of Initial Motion.**—If we substitute $\dot{\xi}_1 dt$ for $\delta\xi_1$, $\dot{\xi}_2 dt$ for $\delta\xi_2$, &c. in (9), we obtain for the initial energy T of a system starting from rest the equation

$$2T = \Sigma(\Xi\dot{\xi}). \qquad (17)$$

Let us now suppose that on a system having ξ_1, &c. as its generalized coordinates constraints are imposed capable of being expressed as in Art. 284 by equations connecting the coordinates $\xi_1, \xi_2, \ldots . \xi_n$. The coordinates are then no longer independent, and if the system be set in motion by impulses Ξ_1, &c., equations (8) no longer hold good, but (9) and (17) remain valid, $\dot{\xi}_1$, &c. being the velocity-components of the actual motion. Also T is the same function of $\dot{\xi}_1$, &c. as it was before the imposition of the constraints, the only difference being that certain relations hold good in the constrained motion connecting these velocity-components.

In order to compare the initial kinetic energies of the system in the unconstrained and constrained motion, let $\dot{\xi}_1$, &c. be the velocity-components corresponding to the former, and $\dot{\xi}_1'$, &c. those corresponding to the latter, then by (17) we have

$$2T_{\dot{\xi}'} = \Sigma(\Xi\dot{\xi}') = \Sigma(p\dot{\xi}'), \text{ also } 2T_{\dot{\xi}} = \Sigma p\dot{\xi}.$$

Substituting in (15), we obtain

$$T_{(\dot{\xi}' - \dot{\xi})} = T_{\dot{\xi}} - T_{\dot{\xi}'}. \qquad (18)$$

This proves Bertrand's Theorem, Art. 199.

297. **Lagrange's Equations of Motion.**—We saw in Art. 294 that

$$\Sigma m\left(\dot{x}\frac{dx}{d\xi_1} + \dot{y}\frac{dy}{d\xi_1} + \dot{z}\frac{dz}{d\xi_1}\right) = \frac{dT}{d\dot{\xi}_1}; \qquad (19)$$

if we differentiate each side of this equation with respect to the time, and substitute for $\frac{d}{dt}\frac{dx}{d\xi_1}$, &c. their values given by (6), we obtain

$$\Sigma m\left(\ddot{x}\frac{dx}{d\xi_1}+\ddot{y}\frac{dy}{d\xi_1}+\ddot{z}\frac{dz}{d\xi_1}\right)+\Sigma m\left(\dot{x}\frac{d\dot{x}}{d\xi_1}+\dot{y}\frac{d\dot{y}}{d\xi_1}+\dot{z}\frac{d\dot{z}}{d\xi_1}\right)=\frac{d}{dt}\frac{dT}{d\dot{\xi}_1};$$

but $\Sigma m\left(\dot{x}\frac{d\dot{x}}{d\xi_1}+\dot{z}\frac{d\dot{y}}{d\xi_1}+\dot{z}\frac{d\dot{z}}{d\xi_1}\right)$ is plainly $\frac{dT}{d\xi_1}$;

hence we have

$$\Sigma m\left(\ddot{x}\frac{dx}{d\xi_1}+\ddot{y}\frac{dy}{d\xi_1}+\ddot{z}\frac{dz}{d\xi_1}\right)=\frac{d}{dt}\frac{dT}{d\dot{\xi}_1}-\frac{dT}{d\xi_1}. \qquad (20)$$

Now in D'Alembert's equation for continuous forces the coefficient of the independent variation $\delta\xi_1$ is,

$$\Sigma m\left(\ddot{x}\frac{dx}{d\xi_1}+\ddot{y}\frac{dy}{d\xi_1}+\ddot{z}\frac{dz}{d\xi_1}\right)-\Sigma\left(X\frac{dx}{d\xi_1}+Y\frac{dy}{d\xi_1}+Z\frac{dz}{d\xi_1}\right).$$

Hence, if we put

$$\Sigma\left(X\frac{dx}{d\xi_1}+Y\frac{dy}{d\xi_1}+Z\frac{dz}{d\xi_1}\right)=\Xi_1, \text{ \&c.},$$

we have, as the equations of motion of any system,

$$\left.\begin{aligned}\frac{d}{dt}\frac{dT}{d\dot{\xi}_1}-\frac{dT}{d\xi_1}&=\Xi_1\\ \frac{d}{dt}\frac{dT}{d\dot{\xi}_2}-\frac{dT}{d\xi_2}&=\Xi_2\\ &\vdots\\ \frac{d}{dt}\frac{dT}{d\dot{\xi}_n}-\frac{dT}{d\xi_n}&=\Xi_n\end{aligned}\right\}. \qquad (21)$$

The work which would be done by the forces of the system against the displacement $\delta\xi_1$ is $-\Xi_1\delta\xi_1$, accordingly Ξ_1, &c. are the generalized components of force tending to alter the coordinates ξ_1, &c. It is to be observed that the

forces X, Y, Z, &c. are not equivalent to the forces Ξ_1, Ξ_2, &c., unless the variations δx, δy, δz, &c., and the corresponding variations $\delta\xi_1$, $\delta\xi_2$, &c., result by orthogonal projection from the same possible displacements of the system.

For a conservative system equations (21) become

$$\left.\begin{aligned} \frac{d}{dt}\frac{dT}{d\dot{\xi}_1} - \frac{dT}{d\xi_1} + \frac{dV}{d\xi_1} &= 0 \\ \frac{d}{dt}\frac{dT}{d\dot{\xi}_2} - \frac{dT}{d\xi_2} + \frac{dV}{d\xi_2} &= 0 \\ \vdots \qquad \vdots \qquad & \vdots \\ \frac{d}{dt}\frac{dT}{d\dot{\xi}_n} - \frac{dT}{d\xi_n} + \frac{dV}{d\xi_n} &= 0 \end{aligned}\right\}. \qquad (22)$$

Equations (21) and (22) were first given by Lagrange, and are known as Lagrange's equations of Motion in Generalized Coordinates.

The proof given above for Lagrange's equations holds good even though the time appear explicitly in the equations which determine the Cartesian in terms of the Generalized coordinates. In this case $x = f(\xi_1, \xi_2, \ldots \xi_n, t)$, &c. $\dot{x}$ then contains the additional term $\frac{df}{dt}$; but the equations

$$\frac{d\dot{x}}{d\dot{\xi}_1} = \frac{dx}{d\xi_1}, \text{ \&c., } \frac{d}{dt}\frac{dx}{d\xi_1} = \frac{d\dot{x}}{d\xi_1}, \text{ \&c.,}$$

are still true, and therefore the proof of Lagrange's equations remains valid.

If we put $L = T - V$, the function L is the difference between the kinetic and potential energies of the system, and is called Lagrange's Function.

Equations (22) may now be written in the form

$$\frac{d}{dt}\frac{dL}{d\dot{\xi}_1} - \frac{dL}{d\xi_1} = 0, \ldots \frac{d}{dt}\frac{dL}{d\dot{\xi}_n} - \frac{dL}{d\xi_n} = 0. \qquad (23)$$

This form of the equation of motion is likewise due to Lagrange.

298. **Deduction of the Equation of Energy.**—If we multiply the first of equations (22) by $\dot{\xi}_1$, the second by $\dot{\xi}_2$, &c., and add, we get

$$\Sigma\dot{\xi}\left(\frac{d}{dt}\,\frac{dT}{d\dot{\xi}} - \frac{dT}{d\xi}\right) + \Sigma\left(\dot{\xi}\,\frac{dV}{d\xi}\right) = 0. \tag{24}$$

Now $$2T = \Sigma\dot{\xi}\,\frac{dT}{d\dot{\xi}};$$

and therefore $$2\,\frac{dT}{dt} = \Sigma\left(\dot{\xi}\,\frac{d}{dt}\,\frac{dT}{d\dot{\xi}} + \ddot{\xi}\,\frac{dT}{d\dot{\xi}}\right),$$

hence

$$\Sigma\dot{\xi}\left(\frac{d}{dt}\,\frac{dT}{d\dot{\xi}} - \frac{dT}{d\xi}\right) = 2\,\frac{dT}{dt} - \Sigma\left(\dot{\xi}\,\frac{dT}{d\xi} + \ddot{\xi}\,\frac{dT}{d\dot{\xi}}\right) = \frac{dT}{dt}, \tag{25}$$

since $$\frac{dT}{dt} = \Sigma\left(\dot{\xi}\,\frac{dT}{d\xi} + \ddot{\xi}\,\frac{dT}{d\dot{\xi}}\right).$$

Substituting in (24), we obtain

$$\frac{dT}{dt} + \frac{dV}{dt} = 0,$$

hence we have

$$T + V = E. \tag{26}$$

299. **Effect of Constraints.**—If a system having n degrees of freedom be subjected to additional constraints capable of being expressed by q equations connecting the coordinates of the form $F = 0$, $G = 0$, &c., we may either select a new system of $n - q$ generalized coordinates, or else

retain the old system, and proceed according to the method of Art. 289. Equations (22) would then become

$$\left.\begin{aligned}
\frac{d}{dt}\frac{dT}{d\dot{\xi}_1} - \frac{dT}{d\xi_1} + \frac{dV}{d\xi_1} &= \lambda\frac{dF}{d\xi_1} + \mu\frac{dG}{d\xi_1} + \&c. \\
\frac{d}{dt}\frac{dT}{d\dot{\xi}_2} - \frac{dT}{d\xi_2} + \frac{dV}{d\xi_2} &= \lambda\frac{dF}{d\xi_2} + \mu\frac{dG}{d\xi_2} + \&c. \\
&\vdots \\
\frac{d}{dt}\frac{dT}{d\dot{\xi}_n} - \frac{dT}{d\xi_n} + \frac{dV}{d\xi_n} &= \lambda\frac{dF}{d\xi_n} + \mu\frac{dG}{d\xi_n} + \&c.
\end{aligned}\right\}. \quad (27)$$

In the case of impulses we may proceed in a similar manner, and still make use of equations (8) or (11), provided we introduce additional terms into Ξ_1', &c. representing the impulses by which the constraints may be replaced. It is plain that both in the case of continuous and also in that of impulsive forces the terms in Lagrange's equations representing the action of the constraints disappear from the equation of energy.

Examples.

1. Determine in polar coordinates the equations of motion of a particle which moves freely in a fixed plane.

Here $T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$, whence

$$\frac{d}{dt}\frac{dT}{d\dot{r}} - \frac{dT}{dr} = m\ddot{r} - mr\dot{\theta}^2, \quad \frac{d}{dt}\frac{dT}{d\dot{\theta}} - \frac{dT}{d\theta} = m\frac{d}{dt}(r^2\dot{\theta}),$$

and the equations of motion are the same as those which would be given by (11) and (12), Art. 28.

2. Determine in polar co-ordinates the general equations of motion of a free particle.

Here $$T = \tfrac{1}{2}m\{\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\},$$

and the equations of motion are

$$m\{\ddot{r} - r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\} = R,$$

$$m\left\{\frac{d}{dt}(r^2\dot{\theta}) - r^2\sin\theta\cos\theta\dot{\phi}^2\right\} = Pr, \quad m\frac{d}{dt}(r^2\sin^2\theta\dot{\phi}) = Qr\sin\theta,$$

where R, P, and Q are the components of the force acting on the particle, along the radius vector from the origin, perpendicular to the radius vector in the meridian of the particle, and at right angles to these two directions.

3. Prove Euler's equations for a body having a fixed point.

The body-axes being the principal axes at the fixed point, the expression for T in terms of θ, ϕ, ψ is given in Ex. 5, Art. 288. Hence

$$\frac{d}{dt}\{C\dot\phi + C\dot\psi\cos\theta\} - (A - B)\sin\phi\cos\phi\,(\dot\theta^2 - \dot\psi^2\sin^2\theta)$$
$$+ (A - B)\sin\theta\,(\cos^2\phi - \sin^2\phi)\,\dot\theta\dot\psi = \Phi.$$

If we substitute ω_3 for $\dot\phi + \dot\psi\cos\theta$ by (12), Art. 258, and then make $\theta = \phi = \frac{1}{2}\pi$, we have

$$C\frac{d\omega_3}{dt} - (A - B)\,\omega_1\,\omega_2 = \Phi = N. \quad \text{(Ex. 4, Art. 288.)}$$

4. Generalize Euler's equations for the case in which the body-axes are not principal axes.

In this case T is a quadratic function of ω_1, ω_2, ω_3, with constant coefficients (Art. 263). Hence, by Ex. 5, Art. 260,

$$\frac{dT}{d\dot\phi} = \frac{dT}{d\omega_1}\frac{d\omega_1}{d\dot\phi} + \frac{dT}{d\omega_2}\frac{d\omega_2}{d\dot\phi} + \frac{dT}{d\omega_3}\frac{d\omega_3}{d\dot\phi} = \frac{dT}{d\omega_3},$$

$$\frac{dT}{d\phi} = \frac{dT}{d\omega_1}\frac{d\omega_1}{d\phi} + \frac{dT}{d\omega_2}\frac{d\omega_2}{d\phi} + \frac{dT}{d\omega_3}\frac{d\omega_3}{d\phi} = \frac{dT}{d\omega_1}\,\omega_2 - \frac{dT}{d\omega_2}\,\omega_1;$$

and we have

$$\frac{d}{dt}\left(\frac{dT}{d\omega_3}\right) - \frac{dT}{d\omega_1}\omega_2 + \frac{dT}{d\omega_2}\,\omega_1 = \Phi = N.$$

5. Two particles m and m' are connected by an inextensible string passing through a smooth hole at the edge of a smooth horizontal table on which m rests; determine the equations of motion of the particles, and the tension of the string.

Let r and θ be the polar coordinates of m with respect to the hole as origin; then

$$2T = (m + m')\,\dot r^2 + mr^2\,\dot\theta^2,$$

and the equations of motion are

$$(m + m')\,\ddot r - mr\,\dot\theta^2 = -\,m'g, \quad \frac{d}{dt}(mr^2\,\dot\theta) = 0.$$

If τ be the tension of the string, and h the value of $mr^2\,\dot\theta$, we have

$$m\ddot r - mr\dot\theta^2 = -\,\tau \text{ (Ex. 1)},$$

whence

$$\tau = \frac{mm'}{m + m'}\left(g + \frac{h^2}{m^2 r^3}\right).$$

6. A smooth particle descends the upper edge of a thin vertical lamina, which is capable of sliding freely down a smooth inclined plane with which

its whole lower ledge is in contact. If the plane of the lamina be perpendicular to the intersection of the inclined plane with the horizon, and the particle and lamina start from rest, determine their position at any time.

Let x be the distance at any time of a point in the base of the lamina from its initial position, ξ the distance which the particle has moved along the edge of the lamina, α the angle which this edge makes with the inclined plane, β the inclination of the latter, m the mass of the particle, and M that of the lamina.

The kinetic energy of the lamina at any time is $\frac{1}{2}M\dot{x}^2$, and that of the particle is

$$\tfrac{1}{2}m\{(\dot{x}+\dot{\xi}\cos\alpha)^2+\dot{\xi}^2\sin^2\alpha\}.$$

Hence

$$2T=(M+m)\dot{x}^2+m\dot{\xi}^2+2m\dot{x}\dot{\xi}\cos\alpha.$$

Again, $$-V=Mgx\sin\beta+mg\{(x\sin\beta+\xi\sin(\alpha+\beta)\},$$

and therefore the equations of motion are

$$(M+m)\ddot{x}+m\ddot{\xi}\cos\alpha=(M+m)g\sin\beta,\quad m(\ddot{\xi}+\ddot{x}\cos\alpha)=mg\sin(\alpha+\beta),$$

whence

$$x=\tfrac{1}{2}gt^2\left\{\sin\beta-\frac{m\sin\alpha\cos\alpha\cos\beta}{M+m\sin^2\alpha}\right\},\quad \xi=\tfrac{1}{2}gt^2\frac{(M+m)\sin\alpha\cos\beta}{M+m\sin^2\alpha}.$$

300. **Ignoration of Coordinates.**—If there be no force tending to alter one or more of the independent variables by which the position of a system is defined; if moreover the expression for the kinetic energy of the system does not contain these variables, but only their differential coefficients; and if the system start from rest; then T may be expressed as a function of the other independent variables and their differential coefficients, and be treated as if these latter variables completely defined the position of the system.

Let ξ_1 be one of the independent variables satisfying the conditions supposed; then, as there is no force tending to alter ξ_1,

$$\frac{d}{dt}\frac{dT}{d\dot{\xi}_1}-\frac{dT}{d\xi_1}=0;\ \text{but}\ \frac{dT}{d\xi_1}=0;\ \text{and therefore}\ \frac{dT}{d\dot{\xi}_1}=\text{constant};$$

also as the system starts from rest, and T is a homogeneous quadratic function of $\dot{\xi}_1, \dot{\xi}_2 \ldots \dot{\xi}_n$, this constant must be zero; hence $\dfrac{dT}{d\dot{\xi}_1}=0$. In like manner, if ξ_2 be another variable satisfying the same conditions, we have $\dfrac{dT}{d\dot{\xi}_2}=0$, and so on.

From the linear equations $\frac{dT}{d\dot{\xi}_1} = 0$, $\frac{dT}{d\dot{\xi}_2} = 0$, &c., $\dot{\xi}_1$, $\dot{\xi}_2$, &c. can be found in terms of the remaining differential coefficients $\dot{\xi}_q, \ldots \dot{\xi}_n$. Thus T becomes a function of $\xi_q \ldots \xi_n$, and of their differential coefficients, that is

$$T = F(\xi_q, \xi_{q+1}, \text{\&c.}, \dot{\xi}_q, \text{\&c.}).$$

If now we regard $\dot{\xi}_q$, $\dot{\xi}_{q+1}$, &c. as completely defining the position of the system, Lagrange's equations are

$$\frac{d}{dt}\frac{dF}{d\dot{\xi}_q} - \frac{dF}{d\xi_q} = \Xi_q, \text{ \&c.};$$

but these equations are true, for

$$\frac{dF}{d\dot{\xi}_q} = \frac{dT}{d\dot{\xi}_q} + \frac{dT}{d\dot{\xi}_1}\frac{d\dot{\xi}_1}{d\dot{\xi}_q} + \frac{dT}{d\dot{\xi}_2}\frac{d\dot{\xi}_2}{d\dot{\xi}_q} + \text{\&c.},$$

$$\frac{dF}{d\xi_q} = \frac{dT}{d\xi_q} + \frac{dT}{d\dot{\xi}_1}\frac{d\dot{\xi}_1}{d\xi_q} + \frac{dT}{d\dot{\xi}_2}\frac{d\dot{\xi}_2}{d\xi_q} + \text{\&c.};$$

whence, as $\frac{dT}{d\dot{\xi}_1} = 0$, $\frac{dT}{d\dot{\xi}_2} = 0$, &c., we have

$$\frac{dF}{d\dot{\xi}_q} = \frac{dT}{d\dot{\xi}_q}, \quad \frac{dF}{d\xi_q} = \frac{dT}{d\xi_q}, \text{ \&c.}$$

The proposition proved above is given by Thomson and Tait (*Natural Philosophy*), and is the simplest case of what they have termed Ignoration of Coordinates.

Examples.

1. A particle descends from rest along one face of a smooth triangular prism which is supported by a smooth horizontal plane. The initial position of the particle lies in the vertical plane containing the centre of inertia of the prism and perpendicular to its edge; determine the motion.

Let x be the horizontal coordinate, in the vertical plane in which the particle moves, of the centre of inertia of the prism, M its mass, m that of the particle,

ξ the distance it has moved at any time along the face of the prism, and α the angle which this face makes with the horizontal plane; then

$$2T = (M+m)\dot{x}^2 + m\dot{\xi}^2 + 2m\dot{x}\dot{\xi}\cos\alpha, \quad V = -mg\xi\sin\alpha;$$

and the equations of motion are

$$(M+m)\ddot{x} + m\ddot{\xi}\cos\alpha = 0, \quad m\ddot{\xi} + m\ddot{x}\cos\alpha = mg\sin\alpha.$$

Hence, as the particle starts from rest,

$$(M+m)\dot{x} = -m\dot{\xi}\cos\alpha, \quad (M+m\sin^2\alpha)\ddot{\xi} = (M+m)g\sin\alpha.$$

The student will observe that if T were expressed by means of the first of these equations as a function of $\dot{\xi}$ alone, and treated as such, the second equation would be obtained directly as Lagrange's equation.

2. In the preceding example, if the face of the prism down which the particle descends be rough, determine the equations of motion.

The force of friction tends merely to stop the relative motion of the particle and prism; hence, F being this force, $F\delta f = -\mu P\delta\xi$, where P is the perpendicular pressure of the particle on the face of the prism. Now $P = m(g\cos\alpha + \ddot{x}\sin\alpha)$, and therefore the equations of motion are

$$(M+m)\ddot{x} + m\ddot{\xi}\cos\alpha = 0,$$

$$m\ddot{\xi} + m\ddot{x}\cos\alpha = mg(\sin\alpha - \mu\cos\alpha) - \mu m\ddot{x}\sin\alpha.$$

The latter of these equations can be reduced to the form

$$\ddot{\xi}\cos\lambda + \ddot{x}\cos(\alpha-\lambda) = g\sin(\alpha-\lambda),$$

where $\tan\lambda = \mu$.

3. A sphere, having no motion of rotation, and under the action of a force passing through its centre of inertia, moves through a liquid extending indefinitely in all directions on one side of an infinite plane: the liquid being originally at rest, and not acted on by any force, determine the form of the equations of motion of the sphere.

Let the origin be anywhere in the fixed plane, the axis of x being at right angles to that plane; and let x, y, z be the coordinates of the centre of the sphere at any time, and ξ a coordinate of any particle of the liquid, which may be defined as matter which is incompressible, devoid of resistance to change of shape, and incapable of exercising any friction against a surface with which it is in contact.

If T be the kinetic energy of the whole system, we have $\dfrac{dT}{d\dot{\xi}} = C$, since there is no force acting on the liquid; but as the liquid was originally at rest, and no impulse was imparted to it, $C = 0$. Hence T is a function of $x, y, z, \dot{x}, \dot{y}, \dot{z}$. Again, the motion of the system at any instant could be produced from rest by placing the sphere in its actual position, and giving it an impulse sufficient to impart to it its actual velocity, since the impulses which should be given to the liquid particles are zero (10), Art. 294. Hence, as the initial circumstances are unaltered by changing the values of y and z, T is a function of $x, \dot{x}, \dot{y}, \dot{z}$. Again, a change in the sign of $\dot{y}$ or $\dot{z}$ can make no change

in the value of T, which must therefore be of the form $\frac{1}{2}\{P\dot{x}^2 + Q(\dot{y}^2 + \dot{z}^2)\}$, since the coefficients of $\dot{x}\dot{y}$, $\dot{y}\dot{z}$, $\dot{z}\dot{x}$ must be zero.

The equations of motion are then

$$Q\ddot{y} + \frac{dQ}{dx}\dot{x}\dot{y} = Y, \quad Q\ddot{z} + \frac{dQ}{dx}\dot{x}\dot{z} = Z,$$

$$P\ddot{x} + \tfrac{1}{2}\left\{\frac{dP}{dx}\dot{x}^2 - \frac{dQ}{dx}(\dot{y}^2 + \dot{z}^2)\right\} = X.$$

4. Prove that a sphere projected through a liquid perpendicularly from an infinite plane boundary is at first accelerated, and then tends towards a constant velocity. Show also that if projected parallel to the boundary it moves as if it were attracted towards the boundary.

Initial circumstances in Ex. 3 are altered in the same manner, whether we suppose introduced into the liquid a second bounding plane parallel to the first, and between it and the sphere, or suppose the sphere placed initially nearer the original bounding plane. Hence a diminution of the initial value of x is equivalent to the introduction of additional geometrical constraints into the system. From this it follows by Bertrand's Theorem, Art. 296, that if $x' < x$, and $P'\dot{x}' = P\dot{x}$, the value of $P\dot{x}^2$ must exceed that of $P'\dot{x}'^2$, and therefore $\dot{x}' < \dot{x}$, and $P' > P$, or P decreases as x increases. Similar reasoning can be applied to Q. If x be infinite, or the liquid unbounded in every direction, P and Q are constants.

The statements made in the enunciation of this example follow then immediately from the equations of Ex. 3, by making X and Y zero.

Examples 3 and 4 are taken, with some slight modifications, from Thomson and Tait (*Natural Philosophy*).

301. **Components of Momentum and Velocities.**—Equations (10), Art. 293, enable us to express the velocities ξ_1, &c. as linear functions of the components of momentum p_1, &c. If these values be substituted for ξ_1, &c. in T, as given by equation (7), a new expression for T is obtained which is a homogeneous quadratic function of $p_1, p_2, \ldots p_n$. We shall represent the two expressions for T by $T_{\dot\xi}$ and T_p. Equation (7), Art. 293 gives $T_{\dot\xi}$, and the corresponding equation for T_p is of the form

$$T_p = P_{11}p_1^2 + P_{22}p_2^2 + \&c. + 2P_{12}p_1p_2 + \&c. \qquad (28)$$

In this equation P_{11}, P_{22}, &c. are functions of ξ_1, ξ_2, &c. Thus $T_{\dot\xi}$ and T_p are each functions of ξ_1, ξ_2, &c.; but these coordinates, so far as they appear explicitly, do not enter in the same manner into the two expressions for T. Equation

(14) gives an expression for T which is symmetrical in $\dot{\xi}_1$ and p_1, &c., and which becomes $T_{\dot{\xi}}$ or T_p according as we express p_1, &c. in terms of $\dot{\xi}_1$, &c., or $\dot{\xi}_1$, &c. in terms of p_1, &c.

If we seek for $\dfrac{dT}{dp_1}$ from equation (14) we obtain

$$2\frac{dT}{dp_1} = \dot{\xi}_1 + \Sigma p \frac{d\dot{\xi}}{dp_1}. \qquad (29)$$

Again, if we seek for $\dfrac{dT}{dp_1}$ from (7) we have

$$\frac{dT}{dp_1} = \frac{dT}{d\dot{\xi}_1}\frac{d\dot{\xi}_1}{dp_1} + \frac{dT}{d\dot{\xi}_2}\frac{d\dot{\xi}_2}{dp_1} + \text{\&c.} = \Sigma p \frac{d\dot{\xi}}{dp_1}. \qquad (30)$$

Substituting this value for $\dfrac{dT}{dp_1}$ in (29) we get

$$2\frac{dT}{dp_1} = \dot{\xi}_1 + \frac{dT}{dp_1}, \text{ whence } \frac{dT}{dp_1} = \dot{\xi}_1;$$

and as a similar result holds good for each component of momentum, we have

$$\frac{dT}{dp_1} = \dot{\xi}_1, \quad \frac{dT}{dp_2} = \dot{\xi}_2, \ldots \frac{dT}{dp_n} = \dot{\xi}_n. \qquad (31)$$

The partial differential coefficients of T with respect to ξ_1, &c. are different according as T is expressed by $T_{\dot{\xi}}$ or T_p. If we seek for $\dfrac{dT_p}{d\xi_1}$ from equation (14) or (7) we must in each case regard $\dot{\xi}_1$, &c. as functions of p_1, p_2, &c.; ξ_1, ξ_2, &c. In this way we get from (14),

$$2\frac{dT_p}{d\xi_1} = p_1\frac{d\dot{\xi}_1}{d\xi_1} + p_2\frac{d\dot{\xi}_2}{d\xi_1} + \text{\&c.}, \qquad (32)$$

and from (7)

$$\frac{dT_p}{d\xi_1} = \frac{dT_{\dot{\xi}}}{d\xi_1} + \frac{dT_{\dot{\xi}}}{d\dot{\xi}_1}\frac{d\dot{\xi}_1}{d\xi_1} + \frac{dT_{\dot{\xi}}}{d\dot{\xi}_1}\frac{d\dot{\xi}_2}{d\xi_1} + \text{\&c.}$$

$$= \frac{dT_{\dot{\xi}}}{d\xi_1} + p_1\frac{d\dot{\xi}_1}{d\xi_1} + p_2\frac{d\dot{\xi}_2}{d\xi_1} + \text{\&c.} \qquad (33)$$

Hence, by (32), $$\frac{dT_p}{d\xi_1} = \frac{dT_{\dot{\xi}}}{d\xi_1} + 2\frac{dT_p}{d\xi_1},$$

and therefore $$\frac{dT_p}{d\xi_1} + \frac{dT_{\dot{\xi}}}{d\xi_1} = 0.$$

We have then the system of equations

$$\frac{dT_p}{d\xi_1} + \frac{dT_{\dot{\xi}}}{d\xi_1} = 0, \quad \frac{dT_p}{d\xi_2} + \frac{dT_{\dot{\xi}}}{d\xi_2} = 0, \ldots \frac{dT_p}{d\xi_n} + \frac{dT_{\dot{\xi}}}{d\xi_n} = 0. \quad (34)$$

It is plain that the reciprocal relations between components of velocity and momentum are analogous to the polar properties of curves and surfaces.

302. **Hamilton's Equations of Motion.**—If we put $T_p + V = U$, we obtain a function U of p_1, p_2, &c., ξ_1, ξ_2, &c., which represents the total energy kinetic and potential of the system, and whose value is constant. By the employment of U Lagrange's equations of motion may be expressed in another very symmetrical form due to Hamilton.

By (10), Art. 294, $\dfrac{d}{dt}\dfrac{dT}{d\dot{\xi}_1} = \dfrac{dp_1}{dt}$, and by (34) $-\dfrac{dT}{d\xi_1} = \dfrac{dT_p}{d\xi_1}$.

Hence Lagrange's equations (22) become

$$\frac{dp_1}{dt} + \frac{dU}{d\xi_1} = 0, \quad \frac{dp_2}{dt} + \frac{dU}{d\xi_2} = 0, \ldots \frac{dp_n}{dt} + \frac{dU}{d\xi_n} = 0. \quad (35)$$

Equations (35) have been termed The Equations of Motion of a system expressed in the Canonical Form.

It is easy to see that the equations which give the motion of the centre of inertia and the changes in the moments of momentum for any system are particular cases of equations (35).

Examples.

1. In a moving system the total elementary change of momentum corresponding to one of the generalized coordinates is made up of two parts, one resulting from the forces acting on the system, the other from the previously existing motion. Show that $\frac{dT}{d\xi}dt$ expresses the latter, ξ being the generalized coordinate.

If p, &c. be the impulses which would give the existing velocities at any instant, $\frac{dT}{d\dot{\xi}} = p$. At the next instant $\left(\frac{dT}{d\dot{\xi}}\right)' = p'$.

From these equations it appears that the total elementary change of momentum $p' - p$ corresponding to ξ is

$$\left(\frac{dT}{d\dot{\xi}}\right)' - \frac{dT}{d\dot{\xi}} \quad \text{or} \quad \frac{d}{dt}\frac{dT}{d\dot{\xi}}dt.$$

Now, by Lagrange's equations,

$$\frac{d}{dt}\frac{dT}{d\dot{\xi}}dt = \Xi dt + \frac{dT}{d\xi}dt,$$

whence, as Ξdt represents the change of momentum resulting from the applied forces, $\frac{dT}{d\xi}dt$ must represent that due to the previous motion.

2. Apply the method of the last example to determine the components of the centrifugal couple in the case of a body having a fixed point.

Here $2T = A\omega_1^2 + B\omega_2^2 + C\omega_3^2$. If now ω_1, ω_2, ω_3 be expressed in terms of $\theta, \phi, \psi;\ \dot{\theta}, \dot{\phi}, \dot{\psi}$,

$$\frac{dT}{d\phi} = \frac{dT}{d\omega_1}\frac{d\omega_1}{d\phi} + \frac{dT}{d\omega_2}\ \frac{d\omega_2}{d\phi} + \frac{dT}{d\omega_3}\ \frac{d\omega_3}{d\phi},$$

then when $\phi = 0$, we have, Art. 258 and Ex. 5, Art. 260,

$$\frac{dT}{d\phi} = (A - B)\,\omega_1\,\omega_2.$$

3. If the Cartesian and generalized coordinates be connected by linear equations with constant coefficients, show that there are no terms in the equations of motion resulting from the previous motion.

303. **Calculus of Variations.**—In the Calculus of Variations the form of the function which determines the dependent variable y in terms of the independent variable x is supposed to vary, and ϖ being the symbol of a given operation or set of operations, the fundamental problem of the Calculus is to determine the variation of ϖy.

If $y = f(x)$, a change whose magnitude is infinitely small in the function $f(x)$ must be of the form $i\psi(x)$, where i is an infinitely small constant. We have then $\delta y = i\psi(x)$. In consequence of y becoming $f(x) + i\psi(x)$, the differential coefficient $\dfrac{d^n y}{dx^n}$ becomes $\dfrac{d^n f}{dx^n} + i\dfrac{d^n \psi}{dx^n}$.

Hence we have
$$\delta \frac{d^n y}{dx^n} = \frac{d^n \delta y}{dx^n}. \tag{36}$$

If
$$\Omega = \int F\left(x, y, \frac{dy}{dx}, \ldots \frac{d^n y}{dx^n}\right) dx,$$

the variation $\delta\Omega$ is the change in Ω in consequence of y changing from $f(x)$ to $f(x) + i\psi(x)$. As the result of this change of y the function F becomes $F + \delta F$, where

$$\delta F = \frac{dF}{dy}\delta y + \frac{dF}{d\left(\dfrac{dy}{dx}\right)}\frac{d\delta y}{dx} \ldots\ldots + \frac{dF}{d\left(\dfrac{d^n y}{dx^n}\right)}\frac{d^n \delta y}{dx^n},$$

and Ω becomes $\int F dx + \int \delta F dx$. Hence we see that

$$\delta\Omega = \delta \int F dx = \int \delta F dx. \tag{37}$$

In the case of a definite integral whose limits are variable the complete variation is the sum of two parts, one resulting from the variation of the limits, the other from the variation of the expression under the integral sign. Hence if $\Omega = \int_{x'}^{x''} F dx$, and if $D\Omega$ be the complete variation of Ω, we have

$$D\Omega = F'' dx'' - F' dx' + \int_{x'}^{x''} \delta F dx. \tag{38}$$

In general the complete variation Du of a dependent variable u is the sum of two parts, one resulting from a change of the independent variable x, the other from a

change in the form of the relation connecting u with x. In the Calculus of Variations the symbol δ is restricted to variations of the latter kind. Hence, in general,

$$Du = \frac{du}{dx} dx + \delta u. \tag{39}$$

EXAMPLES.

1. A particle under the action of gravity is constrained to move from one given point A to another B along a smooth plane curve; determine the nature of the curve so that the time of descent may be the least possible.

The curve obviously lies in a vertical plane passing through the points A and B.

Let the axes of x and y be a vertical and horizontal line in this plane, the positive direction of x being downwards, and let v be the velocity of the particle in any position, then, if the origin be properly selected,

$$v^2 = 2gx, \text{ and therefore } dt = \frac{ds}{\sqrt{2gx}}.$$

Hence, if $$\Omega = \int_{x_0}^{x_1} \sqrt{\frac{1+p^2}{2gx}}\, dx, \quad \text{where} \quad p = \frac{dy}{dx},$$

we have to determine y as a function of x so that Ω may be a minimum, and therefore $\delta\Omega = 0$ for all possible variations of y. Now

$$\delta\Omega = \int_{x_0}^{x_1} \frac{p}{\sqrt{2gx\,(1+p^2)}} \frac{d\delta y}{dx}\, dx;$$

hence, integrating by parts, and neglecting the terms outside the integral sign, since y_1 and y_0 are given, and therefore $\delta y_1 = \delta y_0 = 0$, we have

$$\int_{x_0}^{x_1} \frac{d}{dx}\left(\frac{p}{\sqrt{2gx\,(1+p^2)}}\right) \delta y dx = 0,$$

but δy being arbitrary, this equation cannot be true for all values of δy, except

$$\frac{d}{dx} \frac{p}{\sqrt{2gx\,(1+p)^2}} = 0.$$

Integrating, we have $$p^2 = 2gc^2 x\,(1+p^2).$$

If we put $\dfrac{1}{2gc^2} = a$, and $p = \tan\theta$, we get $x = a\sin^2\theta$, $\dfrac{dy}{dx} = \tan\theta$.

Again $\dfrac{dy}{d\theta} = \dfrac{dy}{dx}\dfrac{dx}{d\theta} = 2a\sin^2\theta$; hence we obtain, as the equations of the curve, $x = a\sin^2\theta$, $y = a(\theta - \sin\theta\cos\theta) + b$, where a and b are arbitrary constants.

The curve is therefore a cycloid (*Differential Calculus*, Art. 272).

This problem is one of great interest in the history of Mathematics, as its proposal by John Bernoulli in 1696 led to the invention of the Calculus of Variations.

2. Prove that for any system of coplanar forces the curve of quickest descent is such that at each point the pressure on the curve due to the forces is equal to that due to the motion.

Here $$\Omega = \int_{x_0}^{x_1} \frac{\sqrt{1+p^2}}{v}\,dx\,;\ \text{hence, putting } \delta\Omega = 0,$$

we have, after integrating by parts, $\dfrac{d}{dx}\left(\dfrac{p}{v\sqrt{1+p^2}}\right) + \dfrac{\sqrt{1+p^2}}{v^2}\,\dfrac{dv}{dy} = 0.$

If we put $p = \tan\theta$, this equation becomes $\dfrac{d}{dx}\left(\dfrac{\sin\theta}{v}\right) + \dfrac{1}{v^2\cos\theta}\,\dfrac{dv}{dy} = 0,$

that is, $$\frac{1}{v}\,\frac{d\sin\theta}{dx} - \frac{\sin\theta}{v^2}\left(\frac{dv}{dx} + \tan\theta\,\frac{dv}{dy}\right) + \frac{1}{v^2\cos\theta}\,\frac{dv}{dy} = 0\,;$$

whence $$\cos\theta\,\frac{d\theta}{dx} = \frac{1}{v}\left(\frac{dv}{dx}\sin\theta - \frac{dv}{dy}\cos\theta\right).$$

Now $\cos\theta = \dfrac{dx}{ds}$, $\sin\theta = \dfrac{dy}{ds}$, and therefore $v^2\dfrac{d\theta}{ds} = v\left(\dfrac{dv}{dx}\,\dfrac{dy}{ds} - \dfrac{dv}{dy}\,\dfrac{dx}{ds}\right)$; also $\dfrac{d\theta}{ds} = \rho$, where ρ is the radius of curvature, and $mv^2 = 2\int(Xdx + Ydy)$; hence, substituting, we obtain

$$\frac{mv^2}{\rho} = X\,\frac{dy}{ds} - Y\,\frac{dx}{ds},$$

which proves the theorem in question.

The curve of quickest descent is called the Brachystochrone. The proposition here proved is a case of a more general theorem in the Calculus of Variations, for the discussion of which the reader is referred to Jellett's *Calculus of Variations*, p. 140, or to the *Encyclopædia Britannica*, vol. 24, p. 86.

3. Deduce Lagrange's equations of motion in generalized coordinates and the corresponding equations for impulses from D'Alembert's Principle by means of the Calculus of Variations.

If x, y, z be the coordinates of any particle m, T is given by the equation $T = \Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$; but T can also be expressed as a function of the generalized coordinates ξ_1, &c., and velocities $\dot{\xi}_1$, &c. As these two expressions for T are always identical, so also are the expressions for $\int \delta T dt$ derived from them; we have therefore

$$\int \Sigma m\left(\dot{x}\,\frac{d\delta x}{dt} + \dot{y}\,\frac{d\delta y}{dt} + \dot{z}\,\frac{d\delta z}{dt}\right) dt = \int\left(\frac{dT}{d\xi_1}\,\delta\xi_1 + \frac{dT}{d\dot{\xi}_1}\,\frac{d\delta\xi_1}{dt} + \text{\&c.}\right) dt.$$

If we integrate by parts each side of this equation, the terms remaining under the integral sign on one side must be equal to those remaining under that sign on the other, and a similar equality must hold good for the terms outside the integral sign at each limit. Hence we have

$$\left(\frac{d}{dt}\,\frac{dT}{d\dot{\xi}_1} - \frac{dT}{d\xi_1}\right)\delta\xi_1 + \left(\frac{d}{dt}\,\frac{dT}{d\dot{\xi}_2} - \frac{dT}{d\xi_2}\right)\delta\xi_2 + \text{\&c.} = \Sigma m(\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z),$$

and $$\frac{dT'}{d\dot{\xi}_1'}\,\delta\xi_1' + \frac{dT'}{d\dot{\xi}_2'}\,\delta\xi_2' + \text{\&c.} = \Sigma m(\dot{x}'\,\delta x' + \dot{y}'\,\delta y' + \dot{z}'\,\delta z').$$

Since the limits are arbitrary the latter equation may be written

$$\frac{dT}{d\dot{\xi}_1}\delta\xi_1 + \frac{dT}{d\dot{\xi}_2}\delta\xi_2 + \&c. = \Sigma m\,(\dot{x}\delta x + \dot{y}\delta y + \dot{z}\delta z).$$

If we now employ D'Alembert's Principle, the equations of motion are immediately obtained.

304. **Least Action.**—The integral $\int 2\,Tdt$ taken between two given configurations of a system is termed the *Action* of the system in passing from one of these configurations to the other. If we denote the action by A, we have the equation

$$A = 2\int_{t'}^{t''} Tdt, \qquad (40)$$

where t' and t'' correspond to the initial and final configurations of the system.

If v be the velocity, m the mass, and s the path of any particle of the system, it is plain that A may be expressed also by the equation

$$A = \Sigma m \int_{s'}^{s''} vds = \Sigma m \int (\dot{x}dx + \dot{y}dy + \dot{z}dz), \qquad (41)$$

where s' and s'' are in any individual motion the values of s for the particle m in the initial and final configurations.

The Principle of Least Action asserts, that subject to the condition imposed by the equation of energy the mode in which a conservative system passes from one configuration to another is such that the action is a minimum.

The equation of energy is $T + V = E$, where E is constant, and V a given function of the coordinates. This equation determines T as a function of the coordinates, but not v the velocity of an individual particle. Hence the value of $\int_{s'}^{s''} vds$ depends not only on the initial and final positions of the particle, but also on the relation which in any individual actual motion exists between v and s. If we consider the expression for A given by (40) it is plain that the value of A depends on the equations which are supposed to determine the coordinates in terms of t in any individual motion of the system, and the Principle of Least Action asserts that in the actual motion of the system these equations are such as to render A a minimum. The student should observe that

in (40) the limiting values of t are not given. In fact, when the initial and final configurations are given the corresponding values of t depend upon the actual motion of the system.

To show that A is a minimum in the actual motion we must suppose the forms of the functions by which x, &c., are expressed in terms of t to vary, and prove that the consequent variation of A is zero.

We have then by (38)

$$DA = 2T''dt'' - 2T'dt' + \int 2\delta T dt.$$

Now $\delta T + \delta V = 0$, and therefore we get

$$DA = 2T''dt'' - 2T'dt' + \int (\delta T - \delta V)\, dt;$$

also, since $\qquad 2T = \Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2),$

we have $\qquad \delta T = \Sigma m(\dot{x}\delta\dot{x} + \dot{y}\delta\dot{y} + \dot{z}\delta\dot{z}),$

hence $$\int \delta T dt = \int \Sigma m\left(\dot{x}\frac{d\delta x}{dt} + \dot{y}\frac{d\delta y}{dt} + \dot{z}\frac{d\delta z}{dt}\right) dt. \tag{42}$$

If we integrate each term by parts, and substitute in the expression for DA, we obtain

$$DA = 2T''dt'' - 2T'dt'$$
$$+ \Sigma m(\dot{x}''\delta x'' + \dot{y}''\delta y'' + \dot{z}''\delta z'') - \Sigma m(\dot{x}'\delta x' + \dot{y}'\delta y' + \dot{z}'\delta z')$$
$$-\int \Sigma\left\{\left(\frac{dV}{dx} + m\ddot{x}\right)\delta x + \left(\frac{dV}{dy} + m\ddot{y}\right)\delta y + \left(\frac{dV}{dz} + m\ddot{z}\right)\delta z\right\} dt. \tag{43}$$

Now by D'Alembert's equation the part under the integral sign must be zero, and therefore if the part outside that sign be likewise zero, we have $DA = 0$.

But $\qquad 2T''dt'' + \Sigma m(\dot{x}''\delta x'' + \dot{y}''\delta y'' + \dot{z}''\delta z'')$

$$= \Sigma m\{\dot{x}''(\dot{x}''dt'' + \delta x'') + \dot{y}''(\dot{y}''dt'' + \delta y'') + \dot{z}''(\dot{z}''dt'' + \delta z'')\},$$

and $\dot{x}''dt'' + \delta x''$, &c. are by (39) the complete variations of x'', &c., and therefore must each be zero, since x'', y'', z'', &c. are invariable, being the coordinates of the particles of the system in its final configuration, which is given. Hence, as similar results hold good for the other limit, we obtain $DA = 0$, and therefore may conclude that A is a minimum or a maximum.

If the potential energy of a system be given as a function of the generalized coordinates, the Principle of Least Action enables us to arrive at its equations of motion.

To obtain the equations of motion in this manner we must seek to determine the generalized coordinates as functions of t in such a way as to make A a minimum, subject to the condition that $T + V =$ constant. This condition gives $\delta T + \delta V = 0$, and therefore if λ be an indeterminate quantity we must have, when A is a minimum,

$$DA + \int \lambda (\delta T + \delta V)\, dt = 0. \tag{44}$$

In this equation the variations $\delta\xi_1$, &c. may be regarded as independent and arbitrary, provided we can determine λ so as to satisfy the equation $T + V =$ constant.

If we substitute $\dfrac{dT}{d\dot{\xi}_1}\,\dfrac{d\delta\xi_1}{dt} + \dfrac{dT}{d\xi_1}\,\delta\xi_1$ + &c. for δT and $\dfrac{dV}{d\xi_1}\,\delta\xi_1$ + &c. for δV in (44), we get, after integrating by parts, for the terms under the sign of integration,

$$\int \left[\left\{ (2+\lambda)\frac{dT}{d\xi_1} + \lambda\frac{dV}{d\xi_1} - \frac{d}{dt}\left[(2+\lambda)\frac{dT}{d\dot{\xi}_1}\right] \right\} \delta\xi_1 + \&\text{c.} \right] dt.$$

Hence, as the part under the integral sign must vanish independently of the terms outside that sign, and as $\delta\xi_1$, &c. are independent and arbitrary, we have the system of equations

$$\left.\begin{aligned} (2+\lambda)\left(\frac{dT}{d\xi_1} - \frac{d}{dt}\frac{dT}{d\dot{\xi}_1}\right) + \lambda\frac{dV}{d\xi_1} - \frac{dT}{d\dot{\xi}_1}\frac{d\lambda}{dt} &= 0 \\ (2+\lambda)\left(\frac{dT}{d\xi_2} - \frac{d}{dt}\frac{dT}{d\dot{\xi}_2}\right) + \lambda\frac{dV}{d\xi_2} - \frac{dT}{d\dot{\xi}_2}\frac{d\lambda}{dt} &= 0 \\ \&\text{c.} \qquad\qquad &= 0 \end{aligned}\right\}. \tag{45}$$

If we multiply the first of these equations by $\dot{\xi}_1$, the second by $\dot{\xi}_2$, &c. and add, we have

$$(2+\lambda)\,\Sigma\left(\frac{dT}{d\xi_1} - \frac{d}{dt}\frac{dT}{d\dot{\xi}}\right)\dot{\xi} + \lambda\Sigma\frac{dV}{d\xi}\dot{\xi} - \frac{d\lambda}{dt}\,\Sigma\dot{\xi}\,\frac{dT}{d\dot{\xi}} = 0. \tag{46}$$

Hence, by (25) and (13), we obtain

$$-(2+\lambda)\frac{dT}{dt}+\lambda\frac{dV}{dt}-2T\frac{d\lambda}{dt}=0,$$

that is,
$$\frac{dT}{dt}-\frac{\lambda}{2+\lambda}\frac{dV}{dt}-\frac{2T}{2+\lambda}\frac{d\lambda}{dt}=0. \tag{47}$$

This equation becomes the same as the equation of condition $T+V=$ constant, provided $\lambda=-(2+\lambda)$, or $\lambda=-1$. Equations (45) then become the same as Lagrange's Equations (22). It is easy to see that if $\lambda=-1$, the terms outside the sign of integration in (44), after integrating by parts, vanish of themselves when the limiting values of ξ_1, ξ_2, &c. are given.

Some eminent mathematicians have deduced the equations of motion from the Principle of Least Action in a strangely illogical manner.

305. **Hamilton's Characteristic Function.**—The motion of a given system having n degrees of freedom whose potential energy is a given function of the coordinates is completely determined if the initial values of the generalized coordinates and velocities be given. At any subsequent undetermined time t we have n equations connecting t with the corresponding values of the coordinates and the $2n$ quantities previously assigned. If t be eliminated from these equations $n-1$ remain. Again, the kinetic and potential energies are at any time connected by the equation $T+V=E$, which gives another relation between the $2n$ assigned quantities. Hence we conclude, that if the initial values of the coordinates be given, and also their values at any subsequent undetermined time, along with the total energy E of the system, the motion is completely determined.

It follows from what has been said that the action A of a system in passing from one configuration to another is a determinate function of the initial and final values of the coordinates and of the total energy. This function is called by Hamilton the Characteristic Function. Whenever it can be assigned it furnishes us with the first and second integrals of the equations of motion, as we proceed to show.

Suppose each of the initial and final coordinates, as well as the total energy of the system, to be slightly altered, then each coordinate, at any intermediate time, receives a corresponding variation, and so likewise does T, the kinetic energy of the system. Now $A = 2\int T dt$, and therefore $\delta A = \int 2\delta T dt$; but $\delta T + \delta V = \delta E$, hence

$$\delta A = \int (\delta T + \delta E - \delta V)\, dt. \qquad (48)$$

If in this we substitute for $\int \delta T dt$ its value given by (42) and integrate by parts, we find, as in (43), that the part under the sign of integration must, in virtue of D'Alembert's equation, be zero. Hence δA must consist entirely of the terms outside the sign of integration. To ascertain what these are when T is expressed as a function of the generalized velocities and coordinates, we must put for δT in (48) the expression

$$\Sigma\left(\frac{dT}{d\xi}\,\delta\xi + \frac{dT}{d\dot{\xi}}\,\frac{d\delta\xi}{dt}\right).$$

Since δA as shown above consists entirely of the terms outside the sign of integration, if ξ_1, ξ_2, &c., ξ_1', ξ_2', &c., be the final and initial coordinates, we obtain thus

$$\delta A = (t-t')\,\delta E + \frac{dT}{d\dot{\xi}_1}\,\delta\xi_1 + \frac{dT}{d\dot{\xi}_2}\,\delta\xi_2 + \&c. - \left(\frac{dT'}{d\dot{\xi}_1'}\,\delta\xi_1' + \frac{dT'}{d\dot{\xi}_2'}\,\delta\xi_2' + \&c.\right).$$

Now $DA = 2Tdt - 2T'dt' + \delta A$, and $2T = \Sigma\,\dfrac{dT}{d\dot{\xi}}\,\dot{\xi}$,

hence by (39) we get

$$DA = (t-t')\,\delta E + p_1 D\xi_1 + p_2 D\xi_2 + \&c. - (p_1' D\xi_1' + p_2' D\xi_2' + \&c.)$$

where p_1, &c. have the same meaning as in (10).

Again, A being supposed to be expressed as a function of the initial and final coordinates and total energy of the system, we have

$$DA = \frac{dA}{d\xi_1} D\xi_1 + \frac{dA}{d\xi_2} D\xi_2 + \&c. + \frac{dA}{d\xi'_1} D\xi_1' + \frac{dA}{d\xi'_2} D\xi_2' + \&c. + \frac{dA}{dE}\,\delta E.$$

Comparing the two expressions for DA, and remembering that $D\xi_1$, $D\xi_2$, &c. $D\xi_1'$, $D\xi_2'$, &c. and δE are independent and arbitrary, we get

$$\frac{dA}{d\xi_1} = p_1, \quad \frac{dA}{d\xi_2} = p_2, \ \ldots \ \frac{dA}{d\xi_n} = p_n; \tag{49}$$

$$\frac{dA}{d\xi_1'} = -p_1', \ \frac{dA}{d\xi_2'} = -p_2', \ \ldots \ \frac{dA}{d\xi_n'} = -p_n'; \tag{50}$$

$$\frac{dA}{dE} = t - t'. \tag{51}$$

Equations (49) and (51), if E be eliminated, furnish expressions for $\dot{\xi}_1$, $\dot{\xi}_2$, &c., in terms of the coordinates and the time, in other words, the first integrals of the equations of motion. Equations (50) and (51), if E be eliminated, enable us to express the coordinates themselves as functions of the time, and so furnish the second integrals of the equations of motion. In each case the initial coordinates ξ_1', &c., and components of momentum p_1', &c., are supposed to be given. It is to be observed that if we desire to have the first integrals in their usual form, in which the arbitrary constants are determined from the initial velocities, we must employ all the equations (49), (50), and (51), and eliminate ξ_1, &c., as well as E.

In the case of a set of *free* particles, equations (49) and (50) become

$$\frac{dA}{dx_1} = m_1\dot{x}_1, \ \frac{dA}{dy_1} = m_1\dot{y}_1, \ \frac{dA}{dz_1} = m_1\dot{z}_1, \ \frac{dA}{dx_2} = m_2\dot{x}_2, \text{ \&c.}; \tag{52}$$

$$\frac{dA}{dx_1'} = -m_1\dot{x}_1', \frac{dA}{dy_1'} = -m_1\dot{y}_1', \frac{dA}{dz_1'} = -m_1\dot{z}_1', \frac{dA}{dx_2'} = -m_2\dot{x}_2', \text{ \&c.} \tag{53}$$

The function A satisfies certain partial differential equations by which it may sometimes be determined. These equations are obtained thus:—Multiply the first of equations (49) by $\dot{\xi}_1$, the second by $\dot{\xi}_2$, &c., and add, and we have

$$\frac{dA}{d\xi_1}\dot{\xi}_1 + \frac{dA}{d\xi_2}\dot{\xi}_2 + \text{\&c.} = 2T = 2(E - V). \tag{54}$$

In like manner, from (50) we get

$$\frac{dA}{d\xi_1'}\dot{\xi}_1' + \frac{dA}{d\xi_2'}\dot{\xi}_2' + \&c. = -2T' = 2(V' - E). \qquad (55)$$

In equation (54) we must remember that $\dot{\xi}_1$, $\dot{\xi}_2$, &c. are supposed to be expressed as functions of p_1, p_2, &c., and thus, finally, as functions of

$$\frac{dA}{d\xi_1}, \ \frac{dA}{d\xi_2}, \ \&c.$$

A similar remark holds good for (55).

In the case of *free* unconnected particles, equations (54) and (55) take the simple forms,

$$\Sigma\frac{1}{m}\left\{\left(\frac{dA}{dx}\right)^2 + \left(\frac{dA}{dy}\right)^2 + \left(\frac{dA}{dz}\right)^2\right\} = 2(E - V). \qquad (56)$$

$$\Sigma\frac{1}{m}\left\{\left(\frac{dA}{dx'}\right)^2 + \left(\frac{dA}{dy'}\right)^2 + \left(\frac{dA}{dz'}\right)^2\right\} = 2(E - V'). \qquad (57)$$

Examples.

1. Find the characteristic function, and the initial and final integrals in the case of a body falling vertically.

Here there is only one coordinate, z the height of the body from the ground. Since gravity tends to diminish z, the potential energy $V = mgz$, and $E = T + mgz$. We have, then,

$$\frac{1}{m}\left(\frac{dA}{dz}\right)^2 = 2(E - mgz), \quad \frac{1}{m}\left(\frac{dA}{dz'}\right)^2 = 2(E - mgz'),$$

where z' is the initial height. If we attribute the negative sign to the square root in the first of these equations, we get, by integrating,

$$A = \frac{m}{3g}\left(\frac{2(E - mgz)}{m}\right)^{\frac{3}{2}} + C.$$

In this equation C is a function of z', and is to be determined from the second differential equation for A. Remembering that A must vanish when $z = z'$, we get finally

$$A = \frac{m}{3g}\left\{\left(\frac{2(E - mgz)}{m}\right)^{\frac{3}{2}} - \left(\frac{2(E - mgz')}{m}\right)^{\frac{3}{2}}\right\}.$$

We have, then,

$$m\dot{z} = p_1 = \frac{dA}{dz} = -m\sqrt{\frac{2(E-mgz)}{m}}, \quad m\dot{z}' = p_1' = -\frac{dA}{dz'} = -m\sqrt{\frac{2(E-mgz')}{m}},$$

$$t = \frac{dA}{dE} = \frac{1}{g}\left\{\left(\frac{2(E-mgz)}{m}\right)^{\frac{1}{2}} - \left(\frac{2(E-mgz')}{m}\right)^{\frac{1}{2}}\right\}.$$

If we eliminate E and z' from these three equations, and put $\dot{z}' = -v'$, we get the ordinary first integral of the equation of motion in which the initial velocity is supposed to be given. If we merely eliminate E between the last two of the above equations, and put $\dot{z}' = -v'$, we get the ordinary final integral.

The resulting equations are $\dot{z} = -(gt + v')$, $z = -g\frac{t^2}{2} - v't + z'$.

The signs which we have attributed to the square roots correspond to the motion of a falling body projected vertically downwards. The results which hold good in the other cases of the motion of a body falling vertically are deduced from the general equations by giving the proper signs to the square roots.

2. A material particle is acted on by an attractive force passing through a fixed point, and varying directly as the distance; find the characteristic function.

Let m be the mass of the particle, and μr the magnitude of the force at the distance r, then

$$-\frac{dV}{dx} = -\mu x, \quad \text{and} \quad V = \frac{\mu}{2}(x^2 + y^2).$$

Hence we have

$$\left(\frac{dA}{dx}\right)^2 + \left(\frac{dA}{dy}\right)^2 = m\{2E - \mu(x^2+y^2)\}. \qquad (a)$$

If we assume

$$\frac{dA}{dx} = \sqrt{mc_1}\sqrt{\left(1 - \frac{\mu}{c_1}x^2\right)}, \quad \frac{dA}{dy} = \sqrt{mc_2}\sqrt{\left(1 - \frac{\mu}{c_2}y^2\right)}, \qquad (b)$$

the equation (a) is satisfied, provided

$$c_1 + c_2 = 2E. \qquad (c)$$

Since the differential equation to be satisfied by $\frac{dA}{dx'}$ and $\frac{dA}{dy'}$ is similar to (a), and since A must vanish when $x = x'$ and $y = y'$, we have

$$A = \frac{\sqrt{m}\,c_1}{2\sqrt{\mu}}\left\{\sqrt{\frac{\mu}{c_1}}\left[x\sqrt{\left(1-\frac{\mu}{c_1}x^2\right)} - x'\sqrt{\left(1-\frac{\mu}{c_1}x'^2\right)}\right]\right.$$

$$\left. + \sin^{-1}\sqrt{\frac{\mu}{c_1}}\,x - \sin^{-1}\sqrt{\frac{\mu}{c_1}}\,x'\right\} + \frac{\sqrt{m}\,c_2}{2\sqrt{\mu}}\left\{\sqrt{\frac{\mu}{c_2}}\left[y\sqrt{\left(1-\frac{\mu}{c_2}y^2\right)}\right.\right.$$

$$\left.\left. - y'\sqrt{\left(1-\frac{\mu}{c_2}y'^2\right)}\right] + \sin^{-1}\sqrt{\frac{\mu}{c_2}}\,y - \sin^{-1}\sqrt{\frac{\mu}{c_2}}\,y'\right\}. \qquad (d)$$

In this expression for A the constants c_1 and c_2 are subject to the condition $c_1 + c_2 = 2E$. In order that A should be expressed as a function of x, y, x', y', and E, a second equation connecting c_1 and c_2 with these quantities is required. This equation is, in fact,

$$\sin^{-1} x\sqrt{\frac{\mu}{c_1}} - \sin^{-1} x'\sqrt{\frac{\mu}{c_1}} = \sin^{-1} y\sqrt{\frac{\mu}{c_2}} - \sin^{-1} y'\sqrt{\frac{\mu}{c_2}}. \qquad (e)$$

Its truth may be proved as follows :—

By equation (d) A is expressed as a function of x, y, x', y', c_1, c_2, so that we may write $A = \phi\,(x, y, x', y', c_1, c_2)$. An equation must exist between c_1, c_2, x, y, x', y', by means of which ϕ can be transformed into $\psi(x, y, x', y', c_1 + c_2)$. We have, then,

$$\frac{d\psi}{dc_1} = \frac{d\phi}{dc_1} + \frac{d\phi}{dc_2}\frac{dc_2}{dc_1}, \quad \frac{d\psi}{dc_2} = \frac{d\phi}{dc_2} + \frac{d\phi}{dc_1}\frac{dc_1}{dc_2}, \text{ but } \frac{d\psi}{dc_1} = \frac{d\psi}{dc_2},$$

and therefore $\dfrac{d\phi}{dc_1}\left(1 - \dfrac{dc_1}{dc_2}\right) = \dfrac{d\phi}{dc_2}\left(1 - \dfrac{dc_2}{dc_1}\right)$, that is, $\dfrac{d\phi}{dc_1}\,dc_1 + \dfrac{d\phi}{dc_2}\,dc_2 = 0.$

Again, $dc_1 + dc_2 = 0$, since $c_1 + c_2 = 2E$, and therefore we have $\dfrac{d\phi}{dc_1} = \dfrac{d\phi}{dc_2}$.

Hence the required relation between c_1 and c_2 must, in virtue of (c), be capable of being expressed in the form, $\dfrac{d\phi}{dc_1} = \dfrac{d\phi}{dc_2}$. The expressions for $\dfrac{d\phi}{dc_1}$ and $\dfrac{d\phi}{dc_2}$ are found most easily from (b). From these equations we have

$$\frac{dA}{dx} = \sqrt{m}\,\sqrt{c_1 - \mu x^2}, \quad \text{whence } \frac{d^2A}{dx\,dc_1} = \frac{\sqrt{m}}{2\sqrt{(c_1 - \mu x^2)}}\,.$$

Integrating, we have

$$\frac{dA}{dc_1} = \tfrac{1}{2}\sqrt{m}\int_{x'}^{x}\frac{dx}{\sqrt{(c_1 - \mu x^2)}} = \tfrac{1}{2}\sqrt{\frac{m}{\mu}}\left(\sin^{-1}x\sqrt{\frac{\mu}{c_1}} - \sin^{-1}x'\sqrt{\frac{\mu}{c_1}}\right).$$

In like manner

$$\frac{dA}{dc_2} = \tfrac{1}{2}\sqrt{\frac{m}{\mu}}\left(\sin^{-1}y\sqrt{\frac{\mu}{c_2}} - \sin^{-1}y'\sqrt{\frac{\mu}{c_2}}\right);$$

hence, since $\dfrac{dA}{dc_1} = \dfrac{dA}{dc_2}$, we have ($e$).

CHAPTER XIII.

SMALL OSCILLATIONS.

306. **Introductory Considerations.**—When a material system in equilibrium under the action of any forces is slightly disturbed, the several points of the system in many cases tend to return to their original positions. In such cases, if the distance of each point from its position of equilibrium remains during the motion very small as compared with the other magnitudes on which the motion depends, the system performs *small oscillations.*

Some cases of small oscillations have been already considered in Articles 102 and 193. The simplification of the problem in the case of small oscillations has been exemplified in the articles referred to, and consists in neglecting the squares and higher powers of small quantities.

Before proceeding to the general theory of small oscillations we shall illustrate the method by the consideration of a few elementary cases.

307. **Oscillation on a Plane Curve.**—We commence with the small oscillation of a particle, under the action of gravity, on a smooth vertical circle.

Taking the lowest point on the circle as origin, the vertical diameter as axis of z, and the tangent as that of x, the equation of the circle is

$$2az = x^2 + z^2, \qquad (1)$$

where a is its radius.

Also, by D'Alembert's principle (Art. 196) we have

$$\ddot{x}\delta x + \ddot{z}\delta z + g\delta z = 0. \qquad (2)$$

Now, for a small oscillation x must be small throughout the motion, and consequently z is a small quantity of the second order.

Hence, to the degree of approximation required, we have

$$a\delta z = x\delta x, \quad \text{and } a\dot{z} = x\dot{x}; \quad \text{therefore } a\ddot{z} = x\ddot{x} + \dot{x}^2;$$

we may accordingly neglect $\ddot{z}$, and equation (2) becomes

$$\left(\ddot{x} + \frac{g}{a}x\right)\delta x = 0, \quad \text{or } \ddot{x} + \frac{g}{a}x = 0.$$

The integral of this equation is

$$x = k \sin\left(t\sqrt{\frac{g}{a}} + \chi\right), \tag{3}$$

as in Art. 102.

In like manner, if any curve be taken instead of the circle, its equation, referred to the tangent and normal at its lowest point, may be written

$$2z = c_0x^2 + 2c_1xz + 2c_2z^2 + \&c.$$

Accordingly, neglecting terms of a higher order than the second, we have $\delta z = c_0x\delta x$, and it is readily seen that $\ddot{z}$ may be neglected as before; also observing that $c_0 = \frac{1}{\rho}$ (*Diff. Cal.*, Art. 230), where ρ is the radius of curvature at the origin, we get immediately from (2),

$$x = k \sin\left(t\sqrt{\frac{g}{\rho}} + \chi\right).$$

This shows that in all such cases the motion is represented by a simple harmonic function.

308. **Oscillation on a Smooth Surface.**—We shall next consider the case of a small oscillation, under gravity, on a smooth spherical surface.

Taking the origin at the lowest point on the sphere, and the z axis vertical, the equation of the sphere is

$$2az = x^2 + y^2 + z^2. \tag{4}$$

Also, from D'Alembert's principle,

$$\ddot{x}\delta x + \ddot{y}\delta y + \ddot{z}\delta z + g\delta z = 0. \tag{5}$$

Here we may neglect z^2 and $\ddot{z}$ as before, and thus we obtain immediately

$$\left(\ddot{x} + \frac{g}{a}x\right)\delta x + \left(\ddot{y} + \frac{g}{a}y\right)\delta y = 0.$$

Hence $$\ddot{x} + \frac{g}{a}x = 0, \quad \ddot{y} + \frac{g}{a}y = 0;$$

accordingly we have

$$x = m\sin\left(t\sqrt{\frac{g}{a}} + \chi_1\right), \quad y = n\sin\left(t\sqrt{\frac{g}{a}} + \chi_2\right),$$

where m, n, χ_1, χ_2 are arbitrary constants.

These equations may also be written in the form

$$\left.\begin{aligned} x &= \alpha\sin t\sqrt{\frac{g}{a}} + \alpha'\cos t\sqrt{\frac{g}{a}} \\ y &= \beta\sin t\sqrt{\frac{g}{a}} + \beta'\cos t\sqrt{\frac{g}{a}} \end{aligned}\right\}, \qquad (6)$$

in which α, β, α', β' are small arbitrary constants, whose values depend on the initial circumstances of the motion.

Hence, if the particle be set in motion with a small initial velocity from a point near the lowest point on a sphere, its motion will be given by equations (6).

Also, if we eliminate t from these equations, we see that the horizontal projection of the path of the particle is an ellipse. (Compare Art. 193.)

We shall now consider the oscillatory motion of a particle, under gravity, on any smooth concave surface.

Neglecting, as in the former cases, small quantities of a higher order than the second, the equation of the surface, when referred to the normal and tangent plane at its lowest point, may be written

$$2z = ax^2 + 2hxy + by^2. \qquad (7)$$

This gives $$\delta z = (ax + hy)\,\delta x + (hx + by)\,\delta y.$$

Also $\ddot{z}$ may be neglected, as before, and equation (5) becomes

$$\{\ddot{x} + g(ax + hy)\}\,\delta x + \{\ddot{y} + g(hx + by)\}\,\delta y = 0.$$

Hence

$$\ddot{x} + g(ax + hy) = 0, \quad \ddot{y} + g(hx + by) = 0. \tag{8}$$

Now, these being linear differential equations, we may put

$$x = m \sin(t\sqrt{\lambda} + \chi), \quad y = n \sin(t\sqrt{\lambda} + \chi);$$

this leads to the equations

$$(ga - \lambda)\,m + ghn = 0, \quad ghm + (gb - \lambda)\,n = 0.$$

Accordingly λ must be a root of the equation

$$\begin{vmatrix} ga - \lambda, & gh \\ gh, & gb - \lambda \end{vmatrix} = 0, \tag{9}$$

and

$$n = \frac{\lambda - ga}{gh}\,m.$$

Hence, if λ_1 and λ_2 be the roots of (9), we see that the complete integrals of (8) may be written

$$\left.\begin{aligned} x &= m_1 \sin(t\sqrt{\lambda_1} + \chi_1) + m_2 \sin(t\sqrt{\lambda_2} + \chi_2) \\ y &= \frac{\lambda_1 - ga}{gh}\,m_1 \sin(t\sqrt{\lambda_1} + \chi_1) + \frac{\lambda_2 - ga}{gh}\,m_2 \sin(t\sqrt{\lambda_2} + \chi_2) \end{aligned}\right\}, \tag{10}$$

in which m_1, m_2, χ_1, χ_2 are arbitrary constants, of which the two former must be very small, in order that the motion should be one of small oscillation. It is readily seen that this solution would fail if either λ_1 or λ_2 were negative; thus if λ be negative, instead of

$$m_1 \sin(t\sqrt{\lambda_1} + \chi_1),$$

we shall have the terms

$$k_1 e^{t\sqrt{\mu_1}} + k'_1 e^{-t\sqrt{\mu_1}},$$

where $\mu_1 = -\lambda_1$.

The motion will then not be a small oscillation, as this expression will increase continually with t, unless in the exceptional case where $k_1 = 0$.

Again, if R_1 and R_2 be the principal radii of curvature at O, it is readily seen that

$$\lambda_1 = \frac{g}{R_1}, \quad \lambda_2 = \frac{g}{R_2}.$$

For let the ellipse $ax^2 + 2hxy + by^2 = c$ be transformed to its axes, so that

$$ax^2 + 2hxy + by^2 = a'X^2 + b'Y^2;$$

then, since $\qquad a + b = a' + b', \quad \text{and} \quad ab - h^2 = a'b',$

the equation $\qquad (ga - \lambda)(gb - \lambda) - g^2h^2 = 0$

becomes $\qquad (ga' - \lambda)(gb' - \lambda) = 0.$

The roots of this equation are ga' and gb'; but, as in Art. 307, we have

$$a' = \frac{1}{R_1}, \quad b' = \frac{1}{R_2},$$

accordingly for a small oscillation both R_1 and R_2 must be positive, *i.e.* the surface must be convex towards the plane of xy. If $\lambda_1 = \lambda_2$, we have $R_1 = R_2$, and the origin is a point of spherical curvature. In this case a small oscillation is the same as on the surface of a sphere, and is given by equations (6).

Examples.

1. A bar of mass m hanging freely from one extremity is slightly displaced; determine its motion.

Take two horizontal lines at right angles to each other passing through the point of suspension for axes of x and y. Let the small angular displacements of the bar at any time round each of these axes towards the other be θ and ϕ; then, r being the distance of any point of the bar from its extremity,

$$x = r\phi, \quad y = r\theta, \quad z^2 = r^2 - x^2 - y^2;$$

therefore $\qquad z = r(1 - \tfrac{1}{2}\theta^2 - \tfrac{1}{2}\phi^2), \quad \delta z = -r(\theta\,\delta\theta + \phi\,\delta\phi),$

thus we may neglect $\ddot{z}$, and have

$$\Sigma dmr^2 (\ddot{\theta}\,\delta\theta + \ddot{\phi}\,\delta\phi) + g\Sigma dmr\,(\theta\,\delta\theta + \phi\,\delta\phi) = 0.$$

Hence, if $\int r^2 dm = mk^2$, $\int r dm = ml$, we have

$$\ddot{\theta} + g\frac{l}{k^2}\theta = 0, \quad \ddot{\phi} + g\frac{l}{k^2}\phi = 0,$$

$$\therefore \theta = \alpha \sin\left(\frac{t}{k}\sqrt{gl} + \chi_1\right), \quad \phi = \beta \sin\left(\frac{t}{k}\sqrt{gl} + \chi_2\right).$$

2. Two balls connected by a horizontal bar, whose mass may be neglected, are suspended by two vertical cords of equal length. The bar receives a slight displacement of rotation round a vertical axis midway between the cords; find the motion of the system.

Let ψ be the angle which the bar makes with a horizontal line parallel to its initial position, θ the inclination of one of the cords to the vertical (see figure in Ex. 4, Art. 244), l its length, b the distance from the middle point of the bar to one of the balls; then

$$x = b\cos\psi = b(1 - \tfrac{1}{2}\psi^2), \quad y = b\psi, \quad z = l(1 - \tfrac{1}{2}\theta^2);$$

but
$$l\theta = b\psi; \quad \therefore z = l\left(1 - \tfrac{1}{2}\frac{b^2}{l^2}\psi^2\right),$$

and
$$x' = -x, \quad y' = -y, \quad z' = z;$$

then $\ddot{x}$, $\ddot{x}'$, $\ddot{z}$, $\ddot{z}'$ may be neglected, and equating to cipher the coefficient of $\delta\psi$ in D'Alembert's equation, we have

$$\ddot{\psi} + \frac{g}{l}\psi = 0;$$

therefore
$$\psi = \alpha \sin\left(\sqrt{\frac{g}{l}}\,t + \chi\right),$$

where α and χ are arbitrary constants.

This shows that the period of vibration is the same as that of the pendulum whose length is l.

3. A heavy bar is suspended and displaced as in the preceding example; investigate its motion.

Let r be the distance of any point of the bar from its centre, and b the distance from its centre to the point of attachment of one of the cords; then, as in the preceding example,

$$\ddot{\psi}\int r^2 dm + g\,\frac{b^2}{l}\,\psi\int dm = 0;$$

therefore
$$\psi = \alpha \sin\left(\frac{b}{k}\sqrt{\frac{g}{l}}\,t + \chi\right), \text{ where } \int r^2 dm = mk^2.$$

4. How must the bar in the preceding example be suspended in order that its vibrations should be isochronous with those of a ball hung by one of the supporting cords?

Ans. $b = k$. In the case of a homogeneous bar whose length is $2a$, $b = \frac{a}{\sqrt{3}}$.

5. A uniform rod of mass m hangs from a horizontal pivot passing through one of its extremities. An inextensible string, whose weight is negligible, attached to the other extremity, passes through a smooth ring situated on the vertical line through the pivot at a distance below it equal to the length of the rod, and sustains a mass p. The rod being slightly displaced from its position of equilibrium, determine the motion.

The equations of motion are

$$\tfrac{4}{3}\, ma^2\, \ddot{\theta} = -\, mga \sin\theta - 2aT, \quad p\, \ddot{z} = T - pg,$$

where a is half the length of the rod, and z the vertical coordinate of p. If z be measured from the position of p when the rod is vertical, $z = 4a \sin \frac{1}{2}\theta$. Since θ is always small, we may take $\sin\theta = \theta$; substituting for $\ddot{z}$ and eliminating T, we have

$$\tfrac{4}{3}\, a(m + 3p)\, \ddot{\theta} + mg\left(\theta + \frac{2p}{m}\right) = 0.$$

Hence the rod returns to its vertical position in a time

$$\sqrt{\left\{\frac{4a}{3g}\left(1 + \frac{3p}{m}\right)\right\}} \cos^{-1}\frac{2p}{2p + m\theta_0},$$

where θ_0 is the initial value of θ.

309. **Stable Equilibrium.**—A position of stable equilibrium is one from which a system has no tendency to depart far if it be slightly disturbed.

In a conservative system if the potential energy be a minimum the corresponding position is one of stable equilibrium; as may be shown in the following manner:—

From equation (4), Art. 282, we have $T + V - V_0 = T_0$. Now since $T = \frac{1}{2}\Sigma mv^2$ it is always essentially positive; also, V_0 being the minimum potential energy, $V - V_0$ is positive for all small values of the variables, and may therefore be reduced to a number of squares with positive coefficients. Therefore if T_0 be small, each term both of T and of $V - V_0$ must be small, and must always remain so. Hence, if the original disturbance be slight the system can never depart far from the position of equilibrium nor attain a high velocity. The position is therefore one of stable equilibrium.

310. Equations of Motion for an Oscillating System.—In the following investigation of the small oscillations of a system about its position of equilibrium, it is assumed that the forces which act at the different points of the system are functions of the coordinates of those points, and that the constraints and mutual connexions can be expressed by means of equations between the coordinates.

In virtue of these equations the coordinates of the points of the system are functions of n independent variables, and these again are at any time functions of their values in the position of equilibrium, and of the increments resulting from the disturbance from this position and subsequent motion. If the system perform small oscillations the increments of the variables are all small quantities, whose squares and higher powers may be neglected.

Hence the equations of motion involve only the first powers of the variables and of their differential coefficients. In other words, they form a system of linear differential equations with constant coefficients.

Let $a_1, a_2, \ldots a_n$ represent the values of the generalized coordinates in the position of equilibrium, and

$$a_1 + \xi_1, \ a_2 + \xi_2, \ \ldots \ a_n + \xi_n$$

their values at any time during the motion. Then x, y, z being the Cartesian coordinates of any point of the system, we have

$$x = f(a_1 + \xi_1, a_2 + \xi_2, \ldots a_n + \xi_n)$$

$$= f(a_1, a_2, \ldots a_n) + \frac{df}{da_1}\xi_1 + \frac{df}{da_2}\xi_2 \ldots + \frac{df}{da_n}\xi_n + \&c., \quad (11)$$

whence, differentiating, we get

$$\dot{x} = \frac{df}{da_1}\dot{\xi}_1 + \frac{df}{da_2}\dot{\xi}_2 \ldots + \frac{df}{da_n}\dot{\xi}_n, \quad (12)$$

since the squares and higher powers of $\dot{\xi}_1$, &c. may be neglected; similar equations hold good for $\dot{y}$ and $\dot{z}$. Hence

T, the kinetic energy of the system, is a quadratic function with *constant coefficients* of $\dot{\xi}_1$, $\dot{\xi}_2$, &c., and we may write

$$2T = f_{11}\dot{\xi}_1^2 + f_{22}\dot{\xi}_2^2 + \text{\&c.} + 2f_{12}\dot{\xi}_1\dot{\xi}_2 + \text{\&c.} \tag{13}$$

Again, if V be the potential energy, we have

$$V = F(a_1 + \xi_1,\ a_2 + \xi_2, \ldots a_n + \xi_n).$$

Expanding by Taylor's Theorem, putting $V_0 = F(a_1, a_2, \ldots a_n)$, and neglecting powers of ξ_1, &c. higher than the second, we get

$$V = V_0 + \frac{dV_0}{da_1}\xi_1 + \frac{dV_0}{da_2}\xi_2 \ldots + \frac{dV_0}{da_n}\xi_n + \tfrac{1}{2}\left(\frac{d^2V_0}{da_1^2}\xi_1^2 + \text{\&c.}\right). \tag{14}$$

Now since V_0 is the potential energy of the system in a position of equilibrium, $\delta V_0 = 0$ for all possible variations of $a_1, a_2, \ldots a_n$, and since these variations are independent and arbitrary, we must have

$$\frac{dV_0}{da_1} = 0,\ \frac{dV_0}{da_2} = 0, \ldots \frac{dV_0}{da_n} = 0. \tag{15}$$

Hence, if we put

$$\frac{d^2V_0}{da_1^2} = q_{11},\ \frac{d^2V_0}{da_2^2} = q_{22},\ \frac{d^2V_0}{da_1 da_2} = q_{12}, \text{ \&c.}, \tag{16}$$

(14) becomes

$$V = V_0 + \tfrac{1}{2}(q_{11}\xi_1^2 + q_{22}\xi_2^2 + 2q_{12}\xi_1\xi_2 + \text{\&c.}) \tag{17}$$

If we substitute the values of T and V given by (13) and (17) in Lagrange's Equations (22) Art. 297, we obtain

$$\left.\begin{array}{l}
f_{11}\ddot{\xi}_1 + f_{12}\ddot{\xi}_2 \ldots + f_{1n}\ddot{\xi}_n + q_{11}\xi_1 + q_{12}\xi_2 \ldots + q_{1n}\xi_n = 0 \\
f_{12}\ddot{\xi}_1 + f_{22}\ddot{\xi}_2 \ldots + f_{2n}\ddot{\xi}_n + q_{12}\xi_1 + q_{22}\xi_2 \ldots + q_{2n}\xi_n = 0 \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
f_{1n}\ddot{\xi}_1 + f_{2n}\ddot{\xi}_2 \ldots + f_{nn}\ddot{\xi}_n + q_{1n}\xi_1 + q_{2n}\xi_2 \ldots + q_{nn}\xi_n = 0
\end{array}\right\}. \tag{18}$$

311. Solution of Equations of Motion—Harmonic Determinant.—As in Art. 308, if we assume

$$\xi_1 = \kappa a_1 \sin(t\sqrt{\lambda} + \chi), \quad \xi_2 = \kappa a_2 \sin(t\sqrt{\lambda} + \chi), \text{ \&c.},$$

and substitute in the equations of motion, we obtain the n equations

$$\left.\begin{array}{l} (f_{11}\lambda - q_{11})\,a_1 + (f_{12}\lambda - q_{12})\,a_2 \ldots + (f_{1n}\lambda - q_{1n})\,a_n = 0 \\ (f_{12}\lambda - q_{12})\,a_1 + (f_{22}\lambda - q_{22})\,a_2 \ldots + (f_{2n}\lambda - q_{2n})\,a_n = 0 \\ \quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ (f_{1n}\lambda - q_{1n})\,a_1 + (f_{2n}\lambda - q_{2n})\,a_2 \ldots + (f_{nn}\lambda - q_{nn})\,a_n = 0 \end{array}\right\}. \quad (19)$$

These can be satisfied by the ratios of the n determinable constants $a_1, a_2, \ldots a_n$, provided λ is a root of the equation

$$\begin{vmatrix} f_{11}\lambda - q_{11}, & f_{12}\lambda - q_{12}, & \ldots & f_{1n}\lambda - q_{1n} \\ f_{12}\lambda - q_{12}, & f_{22}\lambda - q_{22}, & \ldots & f_{2n}\lambda - q_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ f_{1n}\lambda - q_{1n}, & f_{2n}\lambda - q_{2n}, & \ldots & f_{nn}\lambda - q_{nn} \end{vmatrix} = 0. \quad (20)$$

The symmetrical determinant which enters into this equation we shall call Δ. It is usually termed the harmonic determinant of the motion.

If the roots of the equation $\Delta = 0$ be all real and positive, and be denoted by $\lambda_1, \lambda_2, \ldots \lambda_n$, the complete values of ξ_1, ξ_2, &c., are given by the equations

$$\left.\begin{array}{l} \xi_1 = \kappa_1 a_{11}\sin(t\sqrt{\lambda_1} + \chi_1) + \kappa_2 a_{12}\sin(t\sqrt{\lambda_2} + \chi_2) \ldots + \kappa_n a_{1n}\sin(t\sqrt{\lambda_n} + \chi_n) \\ \xi_2 = \kappa_1 a_{21}\sin(t\sqrt{\lambda_1} + \chi_1) + \kappa_2 a_{22}\sin(t\sqrt{\lambda_2} + \chi_2) \ldots + \kappa_n a_{2n}\sin(t\sqrt{\lambda_n} + \chi_n) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \xi_n = \kappa_1 a_{n1}\sin(t\sqrt{\lambda_1} + \chi_1) + \kappa_2 a_{n2}\sin(t\sqrt{\lambda_2} + \chi_2) \ldots + \kappa_n a_{nn}\sin(t\sqrt{\lambda_n} + \chi_n) \end{array}\right\}, \quad (21)$$

where κ_1, χ_1, κ_2, χ_2, &c. are arbitrary constants, $2n$ in number, and $a_{11}, a_{21}, \ldots a_{n1}$ satisfy the n linear equations for $a_1, a_2, \ldots a_n$ obtained by putting λ_1 for λ in (19); $a_{12}, a_{22}, \ldots a_{n2}$ those obtained by putting λ_2 for λ; and so on.

If any root of the equation $\Delta = 0$ be real and negative, instead of $\kappa_1 a_{11} \sin (t\sqrt{\lambda_1} + \chi_1)$, there will be in ξ_1 a term of the form

$$a_{11}\{\kappa_1 e^{t\sqrt{\mu_1}} + \kappa_1' e^{-t\sqrt{\mu_1}}\},$$

where $\mu_1 = -\lambda_1$; and there will be corresponding terms in ξ_2, ξ_3, &c. In fact if we substitute $\kappa_1 a_1 e^{t\sqrt{\mu}}$ for ξ_1, $\kappa_1 a_2 e^{t\sqrt{\mu}}$ for ξ_2, and so on in the equations of motion, we get a system of equations which differ from (19) in having $-\mu$ instead of λ, and which can therefore be satisfied by $a_1 : a_2$, &c. provided $-\mu$ be a root of the equation $\Delta = 0$. Corresponding therefore to every real negative value of λ there is a real positive value of μ. In this case, since ξ_1, ξ_2, &c. contain in general terms increasing without limit with the time, the motion cannot consist of small oscillations.

If we suppose $a_1, a_2, \ldots a_n$ substituted for $\dot{\xi}_1, \dot{\xi}_2, \ldots \dot{\xi}_n$ in T, and for $\xi_1, \xi_2, \ldots \xi_n$ in V, and denote the results of these substitutions by T' and V', equations (19) may be written

$$\frac{d}{da_1}(\lambda T' - V') = 0,\ \frac{d}{da_2}(\lambda T' - V') = 0, \ldots \frac{d}{da_n}(\lambda T' - V') = 0. \quad (22)$$

312. **Lemma in the Theory of Determinants.**—If Δ be any determinant, and if the determinants obtained by erasing the first row and first column of Δ, the second row and second column, the first row and second column, the second row and first column, be denoted by Δ_{11}, Δ_{22}, $-\Delta_{12}$, $-\Delta_{21}$, and if also the determinant formed by erasing the first row and first column of Δ_{11} be denoted by Δ_{1122}, then it is a well-known property of determinants that

$$\Delta_{11}\,\Delta_{22} - \Delta_{12}\,\Delta_{21} = \Delta\Delta_{1122}. \quad (23)$$

For the convenience of the student we shall give here a proof of this theorem.

If we have the n linear equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \ldots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \ldots + a_{2n}x_n = y_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \ldots + a_{3n}x_n = y_3 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 \ldots + a_{nn}x_n = y_n \end{array}\right\}, \qquad (24)$$

and solve for x_1, &c., in terms of y_1, &c., we get another system of n equations, of which the first two are

$$\Delta x_1 = \Delta_{11}y_1 + \Delta_{21}y_2 + \&c.,$$
$$\Delta x_2 = \Delta_{12}y_1 + \Delta_{22}y_2 + \&c.;$$

whence, eliminating y_2, we obtain between the $n + 1$ variables $x_1, x_2, y_1, y_3, \ldots y_n$ the linear equation

$$\Delta(\Delta_{22}x_1 - \Delta_{21}x_2) = (\Delta_{11}\,\Delta_{22} - \Delta_{12}\Delta_{21})y_1 + \&c. \qquad (25)$$

Again we may obtain a linear equation between the same variables in another way, viz., by eliminating $x_3, x_4, \ldots x_n$ from the $(n - 1)$ equations got from equations (24) by omitting the second. The result of this elimination is

$$\begin{vmatrix} a_{11}x_1 + a_{12}x_2 - y_1, & a_{13}, & a_{14}, \ldots a_{1n} \\ a_{31}x_1 + a_{32}x_2 - y_3, & a_{33}, & a_{34}, \ldots a_{3n} \\ \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 - y_n, & a_{n3}, & a_{n4}, \ldots a_{nn} \end{vmatrix} = 0, \qquad (26)$$

which expanded becomes

$$\Delta_{22}x_1 - \Delta_{21}x_2 = \Delta_{1122}y_1 + \&c. \qquad (27)$$

Since only one linear equation can exist between $n + 1$ variables of which n are independent, (27), when multiplied by Δ, must be identical with (25). Hence we have

$$\Delta\Delta_{1122} = \Delta_{11}\Delta_{22} - \Delta_{12}\Delta_{21}.$$

In the case of the harmonic determinant, since it is symmetrical, we have $\Delta_{21} = \Delta_{12}$, and therefore (23) becomes

$$\Delta_{11}\Delta_{22} - \Delta_{12}^2 = \Delta\Delta_{1122}. \qquad (28)$$

313. **Transformation of the Harmonic Determinant.**—If we denote the quadratic function of n variables

$$\tfrac{1}{2}(f_{11}\xi_1^2 + f_{22}\xi_2^2 + 2f_{12}\xi_1\xi_2 + \&c.),$$

by $\mathcal{T}$ and the function

$$\tfrac{1}{2}(q_{11}\xi_1^2 + q_{22}\xi_2^2 + 2q_{12}\xi_1\xi_2 + \&c.),$$

by $\mathfrak{S}$, the harmonic determinant Δ is the discriminant of $\lambda\mathcal{T} - \mathfrak{S}$, and the equation $\Delta = 0$ is therefore unaltered by linear transformation of the variables in $\mathcal{T}$ and $\mathfrak{S}$.

Again, when $\dot{\xi}_1$, $\dot{\xi}_2$, &c., are substituted for the variables in $\mathcal{T}$ it becomes the kinetic energy T of the system. Now, $\dot{\xi}_1$, $\dot{\xi}_2$, being generalized components of velocity, whatever small values be assigned to them, these values will belong to a possible motion of the system. Hence the quantic $\mathcal{T}$ is positive for all real values of the variables, and may therefore be transformed into the sum of n positive squares. If this transformation be effected we have

$$2\mathcal{T} = \eta_1^2 + \eta_2^2 + \eta_3^2 \ldots + \eta_n^2, \tag{29}$$

$$2\mathfrak{S} = s_{11}\eta_1^2 + s_{22}\eta_2^2 + 2s_{12}\eta_1\eta_2 + \&c., \tag{30}$$

and the harmonic determinant is given then by the equation

$$\Delta = \begin{vmatrix} \lambda - s_{11}, & -s_{12}, & -s_{13}, & \ldots - s_{1n} \\ -s_{12}, & \lambda - s_{22}, & -s_{23}, & \ldots - s_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -s_{1n}, & -s_{2n}, & -s_{3n}, & \ldots \lambda - s_{nn} \end{vmatrix}. \tag{31}$$

314. **Reality of the Roots of the Harmonic Determinant Equation.**—If the first row and first column of the harmonic determinant be erased, and a similar process be applied to the determinant so obtained, and again to the determinant thus formed from it, and so on, we get a series of determinants beginning with the harmonic determinant

itself, whose degrees in λ are n, $n-1$, $n-2$, &c., and which in the present Article will be denoted by $\Delta_n, \Delta_{n-1}, \ldots \Delta_1$.

It is to be observed that Δ_n, Δ_{n-1}, Δ_{n-2} are identical with Δ, Δ_{11}, Δ_{1122}, and that Δ_1 is simply $\lambda - s_{nn}$. If we place $+1$ at the end of this series of determinants we obtain a set of $(n+1)$ quantities, such that when any one intermediate between the first and the last vanishes, the two on each side of it take opposite signs. When Δ_{n-1} (that is Δ_{11}) vanishes this appears from (28), and it is plain that a similar equation holds good for any three successive determinants in the series. Its last three terms are,

$$\begin{vmatrix} \lambda - s_{(n-1)(n-1)}, & -s_{(n-1)n} \\ -s_{(n-1)n}, & \lambda - s_{nn} \end{vmatrix}, \quad \lambda - s_{nn}, \quad 1,$$

of which the first is negative when $\lambda - s_{nn} = 0$.

If now we substitute $+\infty$ for λ, each term in the series is positive, and if we substitute $-\infty$ the terms are alternately positive and negative. Hence n variations of sign in the successive terms of the series have been gained in the process of diminishing λ from $+\infty$ to $-\infty$; but, since when one of the intermediate terms vanishes no variation is lost or gained, a variation can be gained only by passing through a root of the equation $\Delta_n = 0$. In this way, therefore, n variations must have been gained. Hence the n roots of the equation $\Delta_n = 0$ are real, and a variation is gained in passing through each.

From this last observation it follows that when Δ_n first vanishes Δ_{n-1} is positive, and that it must become negative before Δ_n vanishes a second time, then again become positive before Δ_n vanishes a third time, and so on. Hence the roots of the equation $\Delta_{n-1} = 0$ separate those of $\Delta_n = 0$. In like manner the roots of the equation $\Delta_{n-2} = 0$ separate those of $\Delta_{n-1} = 0$, and so on.

If we denote by ${}_{\prime}\mathcal{T}$ and ${}_{\prime}\mathfrak{S}$ the quantics obtained, from $\mathcal{T}$ and $\mathfrak{S}$, by omitting all terms containing ξ_1 the minor determinant Δ_{n-1} belonging to Δ_n in its most general form, as written in equation (20), is the discriminant of $\lambda\,{}_{\prime}\mathcal{T} - {}_{\prime}\mathfrak{S}$, and the special form of Δ_{n-1}, considered in this Article, is the

discriminant of the same quantic after linear transformation. Hence the general and special forms of Δ_{n-1} vanish for the same values of λ, and we conclude that in general the roots of the equation $\Delta_{n-1} = 0$ separate those of $\Delta_n = 0$. It is obvious that similar considerations apply in the case of Δ_{n-2}, &c.

The results in this Article might have been obtained directly for the determinants Δ_n, Δ_{n-1}, &c., in their most general form by using the conditions which must be fulfilled (*Diff. Calc.*, p. 460) when the quantic $\mathcal{T}$ is always positive.

315. **Stability of the Motion.**—If we make λ zero in the series of determinants Δ_n, Δ_{n-1}, &c. of Art. 314, we obtain a new series which may be denoted by $(-1)^n D_n$, $(-1)^{n-1} D_{n-1}$, &c., where D_n is the discriminant of $\mathfrak{S}$, and the remaining determinants, D_{n-1}, &c. are formed from D_n by a process similar to that employed in obtaining the former series.

It is clear, from Art. 314, that the number of positive roots of the equation $\Delta_n = 0$ is equal to the number of variations of sign in the successive terms of the series $(-1)^n D_n$, $(-1)^{n-1} D_{n-1}, \ldots - D_1, 1$.

Hence it follows that if D_n, D_{n-1}, &c. be all positive, the harmonic determinant equation has n positive roots. We conclude, therefore (*Diff. Calc.*, p. 460), that in order that the roots of this equation should be all positive, the quantic $\mathfrak{S}$ must be positive for all values of the variables, and *vice versa.*

Without assuming the truth of the conditions referred to, we may obtain the same result in another way by employing the following transformation :—

We shall suppose that $\mathcal{T}$ and $\mathfrak{S}$ are of the form given by equations (29) and (30), and that the roots of the equations $\Delta = 0$ are all unequal.

Apply a linear transformation which will change

$$\eta_1^2 + \eta_2^2 + \eta_3^2 + \text{\&c. into } \zeta_1^2 + \zeta_2^2 + \zeta_3^2 + \text{\&c.},$$

and at the same time reduce $\mathfrak{S}$ to its canonical form

$$P_1 \zeta_1^2 + P_2 \zeta_2^2 \ldots + P_n \zeta_n^2.$$

In order to show that it is possible to do this by a real transformation, assume

$$\left.\begin{array}{l} \eta_1 = \eta_1'\zeta_1 + \eta_1''\zeta_2 + \eta_1'''\zeta_3 + \&c. \\ \eta_2 = \eta_2'\zeta_1 + \eta_2''\zeta_2 + \eta_2'''\zeta_3 + \&c. \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\ \eta_n = \eta_n'\zeta_1 + \eta_n''\zeta_2 + \eta_n'''\zeta_3 + \&c. \end{array}\right\}, \quad (32)$$

where the ratios $\eta_1' : \eta_2' : \eta_3' : \&c.$, are determined by the equations

$$\left.\begin{array}{l} s_{11}\eta_1' + s_{12}\eta_2' + s_{13}\eta_3' \ . \ . \ . + s_{1n}\eta_n' = \lambda_1\eta_1' \\ s_{12}\eta_1' + s_{22}\eta_2' + s_{23}\eta_3' \ . \ . \ . + s_{2n}\eta_n' = \lambda_1\eta_2' \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\ s_{1n}\eta_1' + s_{2n}\eta_2' + s_{3n}\eta_3' \ . \ . \ . + s_{nn}\eta_n' = \lambda_1\eta_n' \end{array}\right\}, \quad (33)$$

the ratios $\eta_1'' : \eta_2'' : \eta_3''' : \&c.$ by the equations

$$\left.\begin{array}{l} s_{11}\eta_1'' + s_{12}\eta_2'' \ . \ . \ . + s_{1n}\eta_n'' = \lambda_2\eta_1'' \\ s_{12}\eta_1'' + s_{22}\eta_2'' \ . \ . \ . + s_{2n}\eta_n'' = \lambda_2\eta_2'' \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\ s_{1n}\eta_1'' + s_{2n}\eta_2'' \ . \ . \ . + s_{nn}\eta_n'' = \lambda_2\eta_n'' \end{array}\right\}, \quad (34)$$

and so on.

From equations (30) and (33) it follows that when λ_1 and λ_2 are unequal,

$$\eta_1'\,\eta_1'' + \eta_2'\,\eta_2'' + \eta_3'\,\eta_3'' \ . \ . \ . + \eta_n'\,\eta_n'' = 0. \quad (35)$$

For $\lambda_1(\eta_1'\,\eta_1'' + \eta_2'\,\eta_2'' + \&c.) = \eta_1''\left(\dfrac{d\mathfrak{S}}{d\eta_1}\right)' + \eta_2''\left(\dfrac{d\mathfrak{S}}{d\eta_2}\right)' + \&c.$

$$= \eta_1'\left(\frac{d\mathfrak{S}}{d\eta_1}\right)'' + \eta_2'\left(\frac{d\mathfrak{S}}{d\eta_2}\right)'' + \&c. = \lambda_2(\eta_1'\,\eta_1'' + \eta_2'\,\eta_2'' + \&c.).$$

We shall now show that $\mathfrak{S}$ becomes of the required form.

In fact $$\frac{d\mathfrak{S}}{d\zeta_1} = \eta_1{}' \frac{d\mathfrak{S}}{d\eta_1} + \eta_2{}' \frac{d\mathfrak{S}}{d\eta_2} \ldots + \eta_n{}' \frac{d\mathfrak{S}}{d\eta_n}$$

$$= \eta_1 \left(\frac{d\mathfrak{S}}{d\eta_1}\right)' + \eta_2 \left(\frac{d\mathfrak{S}}{d\eta_2}\right)' + \&c. = \lambda_1 (\eta_1{}'\eta_1 + \eta_2{}'\eta_2 \ldots + \eta_n{}'\eta_n)$$

$$= \lambda_1 \{(\eta_1{}'^2 + \eta_2{}'^2 + \eta_3{}'^2 + \&c.)\, \zeta_1 + (\eta_1{}'\eta_1{}'' + \eta_2{}'\eta_2{}'' + \&c.)\, \zeta_2 + \&c.\}$$

$$= \lambda_1 (\eta_1{}'^2 + \eta_2{}'^2 + \eta_3{}'^2 \ldots + \eta_n{}'^2)\, \zeta_1, \text{ by (35).}$$

It can be shown in a similar manner that

$$\frac{d\mathfrak{S}}{d\zeta_2} = \lambda_2 (\eta_1{}''^2 + \eta_2{}''^2 \ldots + \eta_n{}''^2)\, \zeta_2, \text{ and so on.}$$

If then we assume, as is allowable,

$$\eta_1{}'^2 + \eta_2{}'^2 + \eta_3{}''^2 \ldots + \eta_n{}'^2 = 1,\ \ \eta_1{}''^2 + \eta_2{}''^2 + \eta_3{}''^2 \ldots + \eta_n{}''^2 = 1, \&c.,$$

we have the equations

$$\frac{d\mathfrak{S}}{d\zeta_1} = \lambda_1\zeta_1,\ \ \frac{d\mathfrak{S}}{d\zeta_2} = \lambda_2\zeta_2, \ \&c.$$

Hence $$2\mathfrak{S} = \lambda_1\zeta_1^2 + \lambda_2\zeta_2^2 \ldots + \lambda_n\zeta_n^2, \qquad (36)$$

and at the same time

$$\eta_1^2 + \eta_2^2 \ldots + \eta_n^2 = \zeta_1^2 + \zeta_2 \ldots + \zeta_n^2. \qquad (37)$$

The constants $\eta_1{}'$, $\eta_2{}'$, &c.; $\eta_1{}''$, $\eta_2{}''$, &c., are obviously the values which a_{11}, a_{21}, &c.; a_{12}, a_{22}, &c. (Art. 311) take in the particular case in which $2\mathcal{T}$ is of the form $\eta_1^2 + \eta_2^2 \ldots + \eta_n^2$.

As the transformation above is real, it follows that if $\mathfrak{S}$ be reduced in *any way* to a form which contains only the squares of the variables, the signs of the coefficients of the different squares are the same as those of λ_1, λ_2, &c.

In order that every term in the general values of ξ_1, ξ_2, &c. should be *periodic*, it is necessary (Art. 311) that all the roots of $\Delta = 0$ should be positive. This condition, as we have just seen, is fulfilled if $\mathfrak{S}$ be reducible to the sum of a number of squares with positive coefficients—in other words, if the function V (Art. 310) be *a minimum for the position of equilibrium*

of the system. In this case, the system being slightly disturbed in any manner from its position of equilibrium has no tendency to depart far from this position, and consequently ξ_1, ξ_2, &c. must remain small throughout the motion. The motion is then stable in its character whatever be the directions of the initial disturbances, and the position for which ξ_1, ξ_2, &c. are zero is one of stable equilibrium.

If V be not a minimum for the position of equilibrium of the system, that is, if some of the coefficients in $\mathfrak{S}$ when reduced to the above form be negative, terms will in general occur in ξ_1, ξ_2, &c. which increase without limit with the time. In this case the position is not one of stable equilibrium, and the motion will not consist of small oscillations, unless the original disturbances be such that the arbitrary constants multiplying terms in ξ_1, &c., which increase without limit with the time, are each zero.

316. **Case of Equal Roots.**—When the equation $\Delta = 0$ has equal roots, the solution in Art. 310 of the differential equations (18) seems to fail from not containing the requisite number of arbitrary constants; and we might suppose that terms containing t as a factor would occur in the values of ξ_1, ξ_2, &c., and therefore that a stable motion of oscillation would not take place for all possible small disturbances. Lagrange and Laplace both fell into this mistake, which was first pointed out by Dr. Routh.

The true theory depends upon the circumstance that when the equation $\Delta = 0$ has a double root λ_1, the system of n linear equations for determining a_1, a_2, &c., Art. 311, are no longer independent, but can be satisfied by $(n - 2)$ of these quantities, the remaining two being arbitrary.

This may be proved as follows:—

If we put

$$\Delta' = \begin{vmatrix} \Lambda_1 - s_{11}, & - s_{12}, & \ldots - s_{1n} \\ - s_{12}, & \Lambda_2 - s_{22}, & \ldots - s_{2n} \\ \vdots & \vdots & \vdots \\ - s_{1n}, & - s_{2n}, & \ldots \Lambda_n - s_{nn} \end{vmatrix},$$

where Λ_1, Λ_2, . . . Λ_n are functions of λ, we have

$$\frac{d\Delta'}{d\lambda} = \frac{d\Delta'}{d\Lambda_1}\frac{d\Lambda_1}{d\lambda} + \frac{d\Delta'}{d\Lambda_2}\frac{d\Lambda_2}{d\lambda} \dots + \frac{d\Delta'}{d\Lambda_n}\frac{d\Lambda_n}{d\lambda}. \tag{38}$$

If we next suppose $\Lambda_1 = \Lambda_2 = \Lambda_3 \dots = \Lambda_n = \lambda$, (38) becomes

$$\frac{d\Delta}{d\lambda} = \Delta_{11} + \Delta_{22} \dots + \Delta_{nn}. \tag{39}$$

Now, from (28) it appears that when Δ vanishes Δ_{11} and Δ_{22} have the same sign, and this holds good for any two of the determinants on the right-hand side of (39); but if λ_1 be a double root of the equation $\Delta = 0$ the right-hand side of (39) must vanish for this value of λ, and as all its terms have the same sign, each must vanish separately. Again, when Δ and Δ_{11} vanish it appears from (28) that Δ_{12} must vanish likewise, and the same is true for every first minor of Δ.

We conclude that when λ is a double root of the equation $\Delta = 0$, the system of n linear equations (19) of Art. 311 can be satisfied by $(n - 2)$ of the quantities $\eta_1', \eta_2' \dots \eta_n'$, the other two remaining arbitrary.

A case of equal roots has been already considered in Art. 308.

We can now show that when two roots, λ_1 and λ_2, are equal, the method already given of effecting the orthogonal transformation still holds good with a slight modification. In fact we have, as before, Art. 315,

$$\eta_1'\,\eta_1''' + \eta_2'\,\eta_2''' + \&c. = 0, \quad \eta_1''\,\eta_1''' + \eta_2''\,\eta_2''' + \&c. = 0,$$
$$\&c. = 0, \qquad \&c. = 0;$$

but in the present case $\eta_1' : \eta_2'$ and $\eta_1'' : \eta_2''$ are both arbitrary, and the two systems $\eta_1', \eta_2', \eta_3'$, &c. and $\eta_1'', \eta_2'', \eta_3''$, &c. differ only in consequence of different values having been assigned to these two arbitrary ratios. By means of one of these ratios we can now satisfy the single equation

$$\eta_1'\,\eta_1'' + \eta_2'\,\eta_2'' \dots + \eta_n'\,\eta_n'' = 0,$$

whilst the other still remains arbitrary. Hence the transformation is complete, but one of the ratios which is determined in the case of unequal roots remains arbitrary in the case of equal.

The results obtained above for the determinants Δ_{11}, Δ_{22}, &c. may be extended, as in Art. 314, to the first minors of Δ in its most general form. We may then assert, in general, that when λ is a double root of the equation $\Delta = 0$, the system of n linear equations (19) can be satisfied by $(n-2)$ of the quantities $a_1, a_2, \ldots a_n$, the other two remaining arbitrary.

The conditions to be fulfilled in the case of equal roots might have been deduced at once from the consideration that the roots of the equation $\Delta_{11} = 0$ separate those of $\Delta = 0$, as shown in Art 314.

If, on the other hand, some method different from that of Art. 314 be adopted to prove the reality of the roots of the equation $\Delta = 0$, then the method of the present Article may be employed to investigate, as above, the case of equal roots, and also to show that the roots of the equation $\Delta_{11} = 0$ separate those of the equation $\Delta = 0$.

317. **General solution of the Differential Equations in the case of Equal Roots.**—When the roots of the equation $\Delta = 0$ are all unequal and positive, equations (21) may be written

$$\left.\begin{array}{l} \xi_1 = a_{11}\sin t\sqrt{\lambda_1} + a'_{11}\cos t\sqrt{\lambda_1} + a_{12}\sin t\sqrt{\lambda_2} + a'_{12}\cos t\sqrt{\lambda_2} + \&c. \\ \xi_2 = a_{21}\sin t\sqrt{\lambda_1} + a'_{21}\cos t\sqrt{\lambda_1} + a_{22}\sin t\sqrt{\lambda_2} + a'_{22}\cos t\sqrt{\lambda_2} + \&c. \\ \&c. = \qquad\qquad \&c. \end{array}\right\}, \quad (40)$$

where the $2n$ constants a_{11}, a'_{11}, a_{12}, a'_{12}, &c., in the expression for ξ_1 are all arbitrary, and the corresponding constants in ξ_2, &c. may be found in terms of these arbitrary constants by the solution of linear equations, the equations connecting $a_{11}, a_{21}, a_{31}, \ldots a_{n1}$ being the same as those connecting $a'_{11}, a'_{21}, a'_{31}, \ldots a'_{n1}$.

If now two roots λ_1 and λ_2 of the equation $\Delta = 0$ become equal, equations (40) are reduced to the form

$$\left.\begin{array}{l} \xi_1 = a_{11}\sin t\sqrt{\lambda_1} + a'_{11}\cos t\sqrt{\lambda_1} + a_{13}\sin t\sqrt{\lambda_3} + a'_{13}\cos t\sqrt{\lambda_3} + \&c. \\ \xi_2 = a_{21}\sin t\sqrt{\lambda_1} + a'_{21}\cos t\sqrt{\lambda_1} + a_{23}\sin t\sqrt{\lambda_3} + a'_{23}\cos t\sqrt{\lambda_3} + \&c. \\ \vdots \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ \xi_n = a_{n1}\sin t\sqrt{\lambda_1} + a'_{n1}\cos t\sqrt{\lambda_1} + a_{n3}\sin t\sqrt{\lambda_3} + a'_{n3}\cos t\sqrt{\lambda_3} + \&c. \end{array}\right\}. \quad (41)$$

In this case there are only $2(n-1)$ arbitrary constants in ξ_1; but since the system of n linear equations corresponding to λ_1 can (Art. 316) be satisfied by $(n-2)$ of the unknown quantities, the other two remaining arbitrary, we may in the present case, in addition to the $(2n-2)$ constants in ξ_1, consider a_{21} and a_{21}' also as arbitrary. We thus have still $2n$ arbitrary constants altogether, and the solution of the differential equations (18) is therefore complete. A particular case of this has been already considered in Art. 308. It is easy to see that we may still, if we please, express the values of ξ_1, &c. by equations (21), but when $\lambda_1 = \lambda_2$ the constants a_{21} and a_{22} are arbitrary, as well as $\kappa_1 a_{11}$, $\kappa_2 a_{12}$, χ_1, and χ_2, and in terms of these six we can express the four arbitrary constants which belong to the solution of the differential equations.

If there be several distinct double roots similar considerations apply to each of them, and in general, corresponding to each double factor of Δ there are four arbitrary constants in the solution of the differential equations.

The preceding investigation can be readily extended to the case in which the equation $\Delta = 0$ has r equal roots.

In this case $2r$ constants $a_{11}, a_{21}, \ldots a_{r1}, a_{11}', a_{21}', \ldots a_{r1}'$ are arbitrary, and the n linear equations corresponding to the multiple root, which in general determine $(n-1)$ quantities in terms of the remaining one, are equivalent to only $(n-r)$ independent equations.

In fact, from what has been proved above, it appears that every double root of the equation $\Delta = 0$ must be a root of $\Delta_{11} = 0$. Hence if the former equation have r equal roots the latter must have $(r-1)$. Again, it is plain that Δ_{11} is related to Δ_{1122} in the same way in which Δ is related to Δ_{11}, and so on. We may therefore conclude that if the equation $\Delta = 0$ have r roots equal to λ_1, then $(r-1)$ successive minors of Δ must vanish for that value of λ.

318. **Principal Coordinates and Directions of Harmonic Vibration.**—Since in the present case the equations are linear which connect different sets of coordinates, the generalized components of velocity are expressed in terms of each other by the same equations as those which connect the corresponding coordinates. Hence

the transformation of coordinates by which $2\mathcal{T}$ becomes $\zeta_1^2 + \zeta_2^2 \ldots + \zeta_n^2$, reduces $2T$ to the form $\dot{\zeta}_1^2 + \dot{\zeta}_2^2 \ldots \dot{\zeta}_n^2$. Now, Art. 315, $2\mathfrak{S}$ is in this case of the form $\lambda_1\zeta_1^2 + \lambda_2\zeta_2^2 \ldots + \lambda_n\zeta_n^2$, and therefore by the solution of the differential equations for this particular set of generalized coordinates we have

$$\zeta_1 = k_1 \sin t(\sqrt{\lambda_1} + \chi_1),\ \zeta_2 = k_2 \sin (t\sqrt{\lambda_2} + \chi_2), \ldots \zeta_n = k_n \sin (t\sqrt{\lambda_n} + \chi_n), \quad (42)$$

where λ_1, &c. are the roots of the equation $\Delta = 0$, and $k_1, k_2, \ldots k_n, \chi_1, \chi_2, \ldots \chi_n$ are arbitrary constants, $2n$ in number.

The coordinates ζ_1, ζ_2, &c. are called the *Principal Coordinates* of the oscillating system.

The Cartesian coordinates x, y, z of any point of the system are given in terms of the principal coordinates by equations of the form

$$\left.\begin{aligned} x &= x_0 + A_1\zeta_1 + A_2\zeta_2 \ldots + A_n\zeta_n \\ y &= y_0 + B_1\zeta_1 + B_2\zeta_2 \ldots + B_n\zeta_n \\ z &= z_0 + C_1\zeta_1 + C_2\zeta_2 \ldots + C_n\zeta_n \end{aligned}\right\}, \quad (43)$$

where x_0, y_0, z_0 are the values of x, y, z respectively for the position of equilibrium, and $A_1, B_1, C_1, A_2, B_2, C_2$, &c. are constants depending (Arts. 310, 315) on the coefficients f_{11}, f_{12}, &c., q_{11}, q_{12}, &c., that is on the connexions between the several particles, and on the forces acting on the system.

From (42) it appears that the motion of each particle is in general resolvable into n simple vibrations whose periods are

$$\frac{2\pi}{\sqrt{\lambda_1}}, \frac{2\pi}{\sqrt{\lambda_2}}, \ldots \frac{2\pi}{\sqrt{\lambda_n}}.$$

The motion of any one particle being determined, that of any other consists of simple vibrations having the same periods, *i. e.* harmonic, with the former.

The direction of motion for the particle xyz, arising from the simple vibration whose period is $\dfrac{2\pi}{\sqrt{\lambda_1}}$, is found by supposing $\zeta_2, \zeta_3, \ldots \zeta_n$ to be each zero, and depends upon the

constants A_1, B_1, C_1. Hence the directions of the several component vibrations, as well as the ratios of their amplitudes for the different particles in any one harmonic set, depend on the particulars of the system, *i.e.* on the connexions and forces; and are independent of the particulars of the motion, *i.e.* of the initial positions and velocities.

The several systems of directions

$$(A_1B_1C_1,\ A_1'B_1'C_1',\ A_1''B_1''C_1'',\ \&c.),$$
$$(A_2B_2C_2,\ A_2'B_2'C_2',\ A_2''B_2''C_2'',\ \&c.),$$
$$\&c.$$

along the constituents of any one of which if the particles xyz, $x'y'z'$, $x''y''z''$, &c. were simultaneously displaced they would all vibrate in the same period or harmonically, are termed the directions of harmonic vibration.

The simple harmonic functions of the time which occur in the expressions for ξ_1, &c. given by equations (21) differ in general only by constant multipliers from the values of ζ_1, ζ_2, &c.

If we put $\kappa_1 \sin(t\sqrt{\lambda_1} + \chi_1) = \psi_1$, $\kappa_2 \sin(t\sqrt{\lambda_2} + \chi_2) = \psi_2$, &c., we may express ψ_1, ψ_2, &c. in terms of ξ_1, ξ_2, &c., as follows:—

Let $2\mathcal{F}_1 = f_{11}a_{11}^2 + f_{22}a_{21}^2 + f_{33}a_{31}^2 + 2f_{12}a_{11}a_{21} + 2f_{13}a_{11}a_{31} + \&c.$

$$2\mathcal{F}_2 = f_{11}a_{12}^2 + f_{22}a_{22}^2 + f_{33}a_{32}^2 + 2f_{12}a_{12}a_{22} + 2f_{13}a_{12}a_{32} + \&c.$$

$$\mathcal{F}_{12} = f_{11}a_{11}a_{12} + f_{22}a_{21}a_{22} + f_{33}a_{31}a_{32} + f_{12}(a_{11}a_{22} + a_{12}a_{21}) + \&c.$$

$$\&c. \qquad \&c. \qquad \&c. \qquad \&c.$$

and let $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_{12}$, &c. denote the expressions obtained by the substitution of q_{11}, q_{12}, &c. for f_{11}, f_{12}, &c. in $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_{12}$, &c. We have, then,

$$2\mathcal{F}_1 = a_{11}\frac{d\mathcal{F}_1}{da_{11}} + a_{21}\frac{d\mathcal{F}_1}{da_{21}} \ldots + a_{n1}\frac{d\mathcal{F}_1}{da_{n1}}, \tag{44}$$

$$2\mathcal{F}_2 = a_{12}\frac{d\mathcal{F}_2}{da_{12}} + a_{22}\frac{d\mathcal{F}_2}{da_{22}} \ldots + a_{n2}\frac{d\mathcal{F}_2}{da_{n2}}, \tag{45}$$

$$\mathcal{F}_{12} = a_{12}\frac{d\mathcal{F}_1}{da_{11}} + a_{22}\frac{d\mathcal{F}_1}{da_{21}} + \&c. = a_{11}\frac{d\mathcal{F}_2}{da_{12}} + a_{21}\frac{d\mathcal{F}_2}{da_{22}} + \&c.; \tag{46}$$

$$\&c. \qquad \&c. \qquad \&c. \qquad \&c$$

and similar equations hold good for $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_{12}$, &c. It is easy to see that $\mathcal{T}_{12}$, $\mathcal{T}_{13}$, &c., $\mathfrak{S}_{12}$, $\mathfrak{S}_{13}$, &c. are each zero. In fact by (22) we have

$$\lambda_1 \frac{d\mathcal{T}_1}{da_{11}} = \frac{d\mathfrak{S}_1}{da_{11}}, \quad \lambda_1 \frac{d\mathcal{T}_1}{da_{21}} = \frac{d\mathfrak{S}_1}{da_{21}}, \quad \lambda_1 \frac{d\mathcal{T}_1}{da_{31}} = \frac{d\mathfrak{S}_1}{da_{31}}, \text{ \&c.} \qquad (47)$$

from which by multiplication and addition we get $\lambda_1\mathcal{T}_{12} = \mathfrak{S}_{12}$. In like manner, $\lambda_2\mathcal{T}_{12} = \mathfrak{S}_{12}$, and therefore in general $\mathcal{T}_{12} = 0$, and $\mathfrak{S}_{12} = 0$.

From (47) we have also $\mathfrak{S}_1 = \lambda_1\mathcal{T}_1$, and in a similar manner $\mathfrak{S}_2 = \lambda_2\mathcal{T}_2$, &c.

If now we multiply the first of equations (21) by $\frac{d\mathcal{T}_1}{da_{11}}$, the second by $\frac{d\mathcal{T}_1}{da_{21}}$, the third by $\frac{d\mathcal{T}_1}{da_{31}}$, and so on, and add, all the simple harmonic functions of the time except ψ_1 disappear. In like manner we can find ψ_2, ψ_3, &c. and thus we obtain

$$\left.\begin{aligned}
2\mathcal{T}_1\psi_1 &= \frac{d\mathcal{T}_1}{da_{11}}\xi_1 + \frac{d\mathcal{T}_1}{da_{21}}\xi_2 \ldots + \frac{d\mathcal{T}_1}{da_{n1}}\xi_n \\
2\mathcal{T}_2\psi_2 &= \frac{d\mathcal{T}_2}{da_{12}}\xi_1 + \frac{d\mathcal{T}_2}{da_{22}}\xi_2 \ldots + \frac{d\mathcal{T}_2}{da_{n2}}\xi_n \\
&\vdots \\
2\mathcal{T}_n\psi_n &= \frac{d\mathcal{T}_n}{da_{1n}}\xi_1 + \frac{d\mathcal{T}_n}{da_{2n}}\xi_2 \ldots + \frac{d\mathcal{T}_n}{da_{nn}}\xi_n
\end{aligned}\right\}. \qquad (48)$$

Since $\mathcal{T}_1$ is a homogeneous quadratic function of a_{11}, a_{21}, &c., and $\mathcal{T}$ the same function of ξ_1, ξ_2, &c., it is plain that the first of equations (48) may be written

$$2\mathcal{T}_1\psi_1 = a_{11}\frac{d\mathcal{T}}{d\xi_1} + a_{21}\frac{d\mathcal{T}}{d\xi_2} \ldots + a_{n1}\frac{d\mathcal{T}}{d\xi_n};$$

now from (21) we have $\frac{d\xi_1}{d\psi_1} = a_{11}$, $\frac{d\xi_2}{d\psi_1} = a_{21}$, &c., and therefore we obtain $2\mathcal{T}_1\psi_1 = \frac{d\mathcal{T}}{d\psi_1}$.

In like manner $2\mathcal{T}_2\psi_2 = \frac{d\mathcal{T}}{d\psi_1}$, &c.; hence we have

$$\mathcal{T} = \mathcal{T}_1\psi_1^2 + \mathcal{T}_2\psi_2^2 \ldots + \mathcal{T}_n\psi_n^2. \qquad (49)$$

In a precisely similar manner we can show that

$$\mathfrak{S} = \mathfrak{S}_1\psi_1^2 + \mathfrak{S}_2\psi_2^2 \ldots + \mathfrak{S}_n\psi_n^2 = \lambda_1\mathcal{T}_1\psi_1^2 + \lambda_2\mathcal{T}_2\psi_2^2 \ldots + \lambda_n\mathcal{T}_n\psi_n^2. \quad (50)$$

If we select the constants a_{11}, a_{12}, a_{13}, ... a_{1n} so as to satisfy the equations $\mathcal{T}_1 = 1$, $\mathcal{T}_2 = 1$, ... $\mathcal{T}_n = 1$, the simple harmonic functions ψ_1, ψ_2, &c. express the values of the principal coordinates of the system.

When the harmonic determinant equation has equal roots the orthogonal transformation which reduces $\mathfrak{S}$ to its canonical form though valid is no longer determinate Art. (316), and there are an indefinite number of sets of principal coordinates.

319. **Effect of Increase of Inertia.**—If the mass or inertia of any part of a moving system be increased, the expression for the kinetic energy receives thereby the addition of one or more terms of the form $\nu\dot{\theta}^2$, where ν is a positive constant, and $\dot{\theta}$ is a linear function of the generalized components of velocity. The coordinates may be transformed in such a way as to make the linear functions θ, &c. identical with an equal number of the generalized coordinates ξ_1, &c.

If the forces acting on the system remain unaltered, and if there be only one additional term in the expression for the kinetic energy, the harmonic determinant Δ' of the system in which there has been an increase of mass or inertia, is given then by the equation

$$\Delta' = \begin{vmatrix} \lambda(f_{11} + \nu) - q_{11}, & \lambda f_{12} - q_{12} & \cdot \\ \lambda f_{12} - q_{12} & \cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot \\ \cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot \\ \cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot \end{vmatrix} = \Delta + \nu\lambda\,\Delta_{11},$$

where Δ is the harmonic determinant of the original system.

If the original position be one of stable equilibrium all the roots λ_1, ... λ_n of the equation $\Delta = 0$ are positive, and are separated by the roots μ_1, ... μ_{n-1} of the equation $\Delta_{11} = 0$. Hence Δ' is positive for $\lambda = \lambda_1$, negative for $\lambda = \mu_1$, negative for $\lambda = \lambda_2$, positive for $\lambda = \mu_2$, and so on. Consequently the roots of the equation $\Delta' = 0$ are each less than the corresponding root of the equation $\Delta = 0$, but are all positive and are separated by the roots of the equation $\Delta_{11} = 0$.

It follows from what has been said, that when the forces remain unaltered an increase of mass increases the several periods of vibration.

If the generalized coordinate θ or ξ_1 were rendered invariable the system would have only $(n - 1)$ degrees of freedom, and the harmonic determinant would become Δ_{11}. Hence no root of the equation $\Delta = 0$ is diminished by an increase of inertia as much as it would be by rendering the corresponding coordinate invariable.

It follows that if any period of oscillation belong to a system both before and after a certain coordinate has been rendered invariable this period belongs also to the system when the mass corresponding to this coordinate is increased.

The substance of this Article is taken from Routh's *Rigid Dynamics.*

320. **Energy of an Oscillating System.**—If we put

$$t\sqrt{\lambda_1} + \chi_1 = \phi_1, \; t\sqrt{\lambda_2} + \chi_2 = \phi_2, \; \text{\&c.},$$

and substitute in T the values of $\dot{\zeta}_1$, $\dot{\zeta}_2$, &c. obtained by differentiating equations (42) we have

$$2T = \lambda_1 k_1^2 \cos^2 \phi_1 + \lambda_2 k_2^2 \cos^2 \phi_2 + \text{\&c.} \tag{51}$$

Again, substituting in V the values of ζ_1, ζ_2, &c. we have

$$2V = 2V_0 + \lambda_1 k_1^2 \sin^2 \phi_1 + \lambda_2 k_2^2 \sin^2 \phi_2 + \text{\&c.} \tag{52}$$

Hence, $$2(T + V) = 2V_0 + \lambda_1 k_1^2 + \lambda_2 k_2^2 \ldots + \lambda_n k_n^2. \tag{53}$$

From equations (51) and (52) it is plain that the sum of the kinetic and potential energies corresponding to each oscillation is constant at each instant and equal to double the *mean value* of either.

It is plain also that the mean value of the total kinetic energy is equal to that of the total potential energy due to the oscillatory motion.

The general expression for the kinetic energy is found by substituting $\dot{\psi}_1$ for ψ_1, $\dot{\psi}_2$ for ψ_2, &c. in (49).
We have thus

$$T = \lambda_1 \kappa_1^2 \mathcal{T}_1 \cos^2\phi_1 + \lambda_2 \kappa_2^2 \mathcal{T}_2 \cos^2\phi_2 \ldots + \lambda_n \kappa_n^2 \mathcal{T}_n \cos^2\phi_n. \tag{54}$$

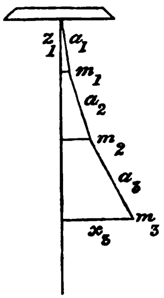
From (50) we obtain

$$V = V_0 + \lambda_1 \kappa_1^2 \mathcal{T}_1 \sin^2\phi_1 + \lambda_2 \kappa_2^2 \mathcal{T}_2 \sin^2\phi_2 \ldots + \lambda_n \kappa_n^2 \mathcal{T}_n \sin^2\phi_n. \tag{55}$$

EXAMPLES.

In the following examples the small oscillations of the system are to be determined in each case :—

1. A number of balls suspended by a fine cord hang in a vertical line, and are slightly displaced in the same vertical plane.



Let x_1, z_1; x_2, z_2, &c. be the horizontal and vertical coordinates of the balls; a_1, a_2, &c. the distances from the point of suspension to the first ball, from the first ball to the second, and so on; θ_1, θ_2, &c. the angles which a_1, a_2, &c. make with the vertical at any instant. The weight of the cord being neglected, the distances a_1, &c. are invariable; then

$$x_1 = a_1\theta_1, \qquad z_1 = a_1 \cos\theta_1 = a_1(1 - \tfrac{1}{2}\theta_1^2),$$

$$x_2 = a_1\theta_1 + a_2\theta_2, \quad z_2 = a_1(1-\tfrac{1}{2}\theta_1^2) + a_2(1-\tfrac{1}{2}\theta_2^2), \text{ \&c.}$$

Substituting in the general dynamical equation, neglecting $\ddot{z}_1$, &c. and equating to zero the coefficients of $\delta\theta_1$, $\delta\theta_2$, &c. we have, after dividing the first equation by a_1, the second by a_2, &c.

$$(m_1 + m_2 + m_3 + \text{\&c.})\, a_1\ddot{\theta}_1 + (m_2 + m_3 + \text{\&c.})\, a_2\ddot{\theta}_2 + (m_3 + \text{\&c.})\, a_3\ddot{\theta}_3 + \text{\&c.} + (m_1 + m_2 + \text{\&c.})\, g\theta_1 = 0,$$

$$(m_2 + m_3 + \text{\&c.})\, a_1\ddot{\theta}_1 + (m_2 + m_3 + \text{\&c.})\, a_2\ddot{\theta}_2 + (m_3 + \text{\&c.})\, a_3\ddot{\theta}_3 + \text{\&c.} + (m_2 + \text{\&c.})\, g\theta_2 = 0,$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$m_n a_1\ddot{\theta}_1 + m_n a_2\ddot{\theta}_2 + m_n a_3\ddot{\theta}_3 + \text{\&c.} + m_n a_n\ddot{\theta}_n + m_n g\theta_n = 0.$$

Hence, assuming

$$\theta_1 = k\alpha \sin(t\sqrt{\lambda} + \chi), \quad \theta_2 = k\beta \sin(t\sqrt{\lambda} + \chi), \text{ \&c.}$$

where k and χ are arbitrary constants, we get to determine α, β, γ, &c., and λ the equations

$$(m_1 + m_2 + \text{\&c.})(a_1\lambda - g)\,\alpha + (m_2 + m_3 + \text{\&c.})\, a_2\lambda\beta + (m_3 + \text{\&c.})\, a_3\lambda\gamma + \text{\&c.} = 0,$$

$$(m_2 + m_3 + \text{\&c.})\, a_1\lambda\alpha + (m_2 + m_3 + \text{\&c.})(a_2\lambda - g)\,\beta + (m_3 + \text{\&c.})\, a_3\lambda\gamma + \text{\&c.} = 0.$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$(a_1\alpha + a_2\beta + a_3\gamma + \text{\&c.} + a_n\omega)\,\lambda - g\omega = 0.$$

This problem can also be treated by the general method of Art. 310. For, since the vertical motion of each ball is very small in comparison with its horizontal motion, the velocities $\dot{z}_1$, $\dot{z}_2$, &c. may be neglected; and we readily find

$$2T = m_1 a_1^2\dot{\theta}_1^2 + m_2(a_1\dot{\theta}_1 + a_2\dot{\theta}_2)^2 + m_3(a_1\dot{\theta}_1 + a_2\dot{\theta}_2 + a_3\dot{\theta}_3)^2$$
$$\ldots + m_n(a_1\dot{\theta}_1 + a_2\dot{\theta}_2 + \ldots + a_n\dot{\theta}_n)^2.$$

Also, if the potential energy be estimated from the position of equilibrium of the system,

$$2V = m_1 g a_1\theta_1^2 + m_2 g(a_1\theta_1^2 + a_2\theta_2^2) + \ldots + m_n g(a_1\theta_1^2 + a_2\theta_2^2 + \ldots + a_n\theta_n^2).$$

The preceding differential equations immediately follow from these equations by the method of Art. 310.

2. The system of balls suspended as in the last example are displaced in different vertical planes.

In this case, θ_1 and ϕ_1 being the angular displacements of a_1 towards the axes of y and x, θ_2 and ϕ_2 those of a_2, &c.

$$x_1 = a_1\phi_1, \quad y_1 = a_1\theta_1, \quad z_1 = a_1(1 - \tfrac{1}{2}\theta_1^2 - \tfrac{1}{2}\phi_1^2), \text{ \&c.};$$

then θ_1, θ_2, &c. are independent of ϕ_1, ϕ_2, &c. as in Ex. 1, Art. 308. The values of θ_1, &c. are therefore the same as in the last example, whilst those of ϕ_1, &c. differ from them only by having a different set of arbitrary constants.

3. A number of rigid bars are hinged together in the same vertical line, and are displaced in the same vertical plane.

If x_1, z_1 be the coordinates of *any* point of the first bar, x_2, z_2 those of any point of the second, &c. and θ_1, θ_2, &c. the inclinations of the bars to the vertical; $x_1 = r_1\theta_1$, $z_1 = r_1(1 - \frac{1}{2}\theta_1^2)$, $x_2 = a_1\theta_1 + r_2\theta_2$, $z_2 = a_1(1 - \frac{1}{2}\theta_1^2) + r_2(1 - \frac{1}{2}\theta_2^2)$, &c.

Proceeding as in Ex. 1, we obtain

$$\{m_1k_1^2 + (m_2 + m_3 + \text{\&c.})\,a_1^2\}\,\ddot{\theta}_1 + \{m_2b_2 + (m_3 + \text{\&c.})\,a_2\}\,a_1\ddot{\theta}_2$$

$$+ \{m_3b_3 + (m_4 + \text{\&c.})\,a_3\}\,a_1\ddot{\theta}_3 + \text{\&c.} + \{m_1b_1 + (m_2 + m_3 + \text{\&c.})\,a_1\}\,g\theta_1 = 0,$$

$$\{m_2b_2 + (m_3 + \text{\&c.})\,a_2\}\,a_1\ddot{\theta}_1 + \{m_2k_2^2 + (m_3 + \text{\&c.})\,a_2^2\}\,\ddot{\theta}_2$$

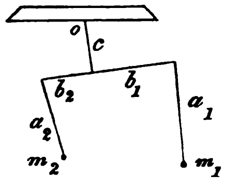
$$+ \{m_3b_3 + (m_4 + \text{\&c.})\,a_3\}\,a_2\ddot{\theta}_3 + \text{\&c.} + \{m_2b_2 + (m_3 + m_4 + \text{\&c.})\,a_2\}\,g\theta_2 = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$m_nb_na_1\ddot{\theta}_1 + m_nb_na_2\ddot{\theta}_2 + m_nb_na_3\ddot{\theta}_3 + \text{\&c.} + m_nk_n^2\ddot{\theta}_n + m_nb_ng\theta_n = 0,$$

where k_1, k_2, &c. are the radii of gyration of the bars round their extremities, and b_1, b_2, &c. the distances from their extremities to their centres of inertia. The solution of these equations is obtained in the same manner as in Ex. 1.

4. Two balls, m_1 and m_2, are suspended by cords, whose lengths are a_1 and a_2, from the extremities of a bar whose position of equilibrium is horizontal, and which hangs from a fixed point by another bar c rigidly attached to the former at a point whose distances from the extremities of the horizontal bar are b_1 and b_2. The system is displaced in the vertical plane which it occupies when in equilibrium.



Let θ_1, θ_2, θ_3, be the inclinations of a_1, a_2, and c to the vertical; x_1, z_1, and x_2, z_2 the coordinates of m_1 and m_2; then

$$x_1 = a_1\theta_1 + b_1(1 - \tfrac{1}{2}\theta_3^2) + c\theta_3, \quad z_1 = a_1(1 - \tfrac{1}{2}\theta_1^2) - b_1\theta_3 + c(1 - \tfrac{1}{2}\theta_3^2),$$

$$x_2 = a_2\theta_2 - b_2(1 - \tfrac{1}{2}\theta_3^2) + c\theta_3, \quad z_2 = a_2(1 - \tfrac{1}{2}\theta_2^2) + b_2\theta_3 + c(1 - \tfrac{1}{2}\theta_3^2).$$

Also let d_1 and d_2 be the distances from O to the extremities of the lever. Then, neglecting small quantities beyond the second order, we readily get

$$2T = m_1(a_1^2\dot{\theta}_1^2 + 2a_1c\dot{\theta}_1\dot{\theta}_3 + d_1^2\dot{\theta}_3^2) + m_2(a_2^2\dot{\theta}_2^2 + 2a_2c\dot{\theta}_2\dot{\theta}_3 + d_2^2\dot{\theta}_3^2) + m_3k_3^2\dot{\theta}_3^2,$$

where $m_3k_3^2$ is the moment of inertia of the lever round the point O.

Also, if th weight of the bar c be neglected,

$$2V = m_1ga_1\theta_1^2 + m_2ga_2\theta_2^2 + Mgc\theta_3^2,$$

where

$$M = m_1 + m_2 + m_3.$$

Hence the differential equations for small oscillatory motion are

$$a_1\ddot{\theta}_1 + c\ddot{\theta}_3 + g\theta_1 = 0, \quad a_2\ddot{\theta}_2 + c\ddot{\theta}_3 + g\theta_2 = 0,$$

and
$$m_1a_1\ddot{\theta}_1 + m_2a_2\ddot{\theta}_2 + Ma_3\ddot{\theta}_3 + Mg\theta_3 = 0,$$

where
$$m_1d_1^2 + m_2d_2^2 + m_3k_3^2 = Ma_3c.$$

If we now assume

$$\theta_1 = \alpha \sin\left(t\sqrt{\frac{g}{\lambda}} + \chi\right), \quad \theta_2 = \beta \sin\left(t\sqrt{\frac{g}{\lambda}} + \chi\right), \quad \theta_3 = \gamma \sin\left(t\sqrt{\frac{g}{\lambda}} + \chi\right),$$

it is readily seen that λ is a root of the cubic

$$M(\lambda - a_1)(\lambda - a_2)(\lambda - a_3) - \lambda(m_1a_1 + m_2a_2)\,c + (m_1 + m_2)\,a_1a_2c = 0.$$

It should be noted that if the cords are equal in length, *i.e.* $a_1 = a_2$, then a_1 is a root of this cubic, and the remaining roots are given by the quadratic

$$M(\lambda - a_1)(\lambda - a_3) - (m_1 + m_2)\,a_1c = 0.$$

This latter furnishes the solution of the small motion of a beam and scales, oscillating in a vertical plane passing through the beam. [See *Camb. Math. Journal*, vol. ii., p. 120.]

5. The balls and cords in the preceding example are replaced by two bars which hang freely from the ends of the lever.

Let l_1 and l_2 be the distances from the ends of the arms of the lever to the centres of inertia of the bars a_1 and a_2, respectively; and let $m_1k_1^2$ be the moment of inertia of the bar a_1 round its upper extremity, and $m_2k_2^2$ the corresponding quantity for a_2; then we readily find

$$2T = m_1(k_1^2\dot{\theta}_1^2 + 2l_1c\dot{\theta}_1\dot{\theta}_3 + d_1^2\dot{\theta}_3^2) + m_2(k_2^2\dot{\theta}_2^2 + 2l_2c\dot{\theta}_2\dot{\theta}_3 + d_2^2\dot{\theta}_3^2) + m_3k_3^2\dot{\theta}_3^2,$$

and, making the potential energy zero in the position of equilibrium, and neglecting the mass of c, we have

$$2V = m_1gl_1\theta_1^2 + m_2gl_2\theta_2^2 + Mgc\theta_3^2.$$

Proceeding as in last example, and putting

$$h_1 = \frac{k_1^2}{l_1}, \quad h_2 = \frac{k_2^2}{l_2},$$

we get, for the determination of λ, the cubic

$$M(\lambda - h_1)(\lambda - h_2)(\lambda - a_3) - \lambda c(m_1l_1 + m_2l_2) + c(m_1l_1h_2 + m_2l_2h_1) = 0.$$

As in the last example, if $h_1 = h_2$, then h_1 is one value of λ, and the other roots of the cubic are those of the quadratic

$$M(\lambda - h_1)(\lambda - a_3) - c(m_1l_1 + m_2l_2) = 0.$$

6. A rigid body, having a fixed point and in stable equilibrium under the action of a conservative system of forces, is slightly disturbed.

Let the axis, a rotation round which would bring the body from its position of equilibrium to its actual position at any time (Art. 249), make angles with the principal axes of the body at the fixed point whose direction cosines are l, m, n. Let σ be the magnitude of the required rotation, which by hypo-

thesis is a small quantity. The coordinates of each point of the body are then at any time functions of constants and of the variables, σ, l, m, n, or, of the three independent variables σl, σm, σn, which may be denoted by θ, ϕ, ψ.

Hence, as θ, ϕ, ψ are small, and the initial position is one of equilibrium (Art. 310),

$$V = V_0 + \tfrac{1}{2}(q_{11}\theta^2 + q_{22}\phi^2 + q_{33}\psi^2 + 2q_{12}\theta\phi + 2q_{13}\theta\psi + 2q_{23}\phi\psi).$$

Again, neglecting small quantities of the second order, $\omega_1 = \dot\theta$, $\omega_2 = \dot\phi$, $\omega_3 = \dot\psi$; and therefore (Art. 263)

$$T = \tfrac{1}{2}(A\dot\theta^2 + B\dot\phi^2 + C\dot\psi^2),$$

neglecting small quantities of the third order.

Hence equations (18) become

$$A\ddot\theta + q_{11}\theta + q_{12}\phi + q_{13}\psi = 0,$$

$$B\ddot\phi + q_{12}\theta + q_{22}\phi + q_{23}\psi = 0,$$

$$C\ddot\psi + q_{13}\theta + q_{23}\phi + q_{33}\psi = 0.$$

Assuming

$$\theta = k\alpha \sin(t\sqrt{\lambda} + \chi), \quad \phi = k\beta \sin(t\sqrt{\lambda} + \chi), \quad \psi = k\gamma \sin(t\sqrt{\lambda} + \chi),$$

we have, for the determination of α, β, γ, λ, the equations

$$A\lambda\alpha = q_{11}\alpha + q_{12}\beta + q_{13}\gamma,$$

$$B\lambda\beta = q_{12}\alpha + q_{22}\beta + q_{23}\gamma,$$

$$C\lambda\gamma = q_{13}\alpha + q_{23}\beta + q_{33}\gamma.$$

If α_1, α_2, &c. be the values of α, &c. corresponding to λ_1 and λ_2, two of the roots of the cubic for λ, it is easy to see that

$$(\lambda_1 - \lambda_2)(A\alpha_1\alpha_2 + B\beta_1\beta_2 + C\gamma_1\gamma_2) = 0;$$

hence

$$A\alpha_1\alpha_2 + B\beta_1\beta_2 + C\gamma_1\gamma_2 = 0,$$

and therefore also

$$\alpha_2(q_{11}\alpha_1 + q_{12}\beta_1 + q_{13}\gamma_1) + \beta_2(q_{12}\alpha_1 + q_{22}\beta_1 + q_{23}\gamma_1)$$
$$+ \gamma_2(q_{13}\alpha_1 + q_{23}\beta_1 + q_{33}\gamma_1) = 0.$$

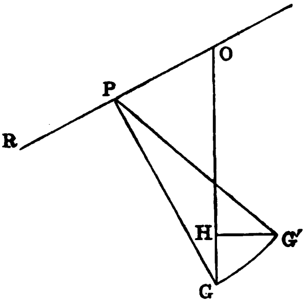
Accordingly the lines whose direction cosines are proportional to α_1, β_1, γ_1; α_2, β_2, γ_2; α_3, β_3, γ_3; are conjugate diameters of the momental ellipsoid, and likewise of the quadric E, whose equation referred to the principal axes of the body at the fixed point is

$$q_{11}x^2 + q_{22}y^2 + q_{33}z^2 + 2q_{12}xy + 2q_{13}xz + 2q_{23}yz = K.$$

Since the initial position is one of stable equilibrium, E must be an ellipsoid (Art. 315).

An angular displacement σ_1 from the position of equilibrium brings the body into a position whose potential energy, relative to the initial position, is $\frac{1}{2}K\left(\frac{\sigma_1}{r_1}\right)^2$, r_1 being the semi-diameter of E round which the rotation σ_1 is effected. Hence all small angular displacements, which are proportional to the diameters of E round which they are effected, bring the body into positions having the same potential energy. On this account Sir Robert Ball calls E the *ellipsoid of equal energy*, or, the *potential ellipsoid* corresponding to the initial position of equilibrium. The results arrived at may be stated as follows:—The *harmonic axes of a rigid body, having a fixed point and in stable equilibrium under the action of a conservative system of forces, are the three conjugate diameters common to the momental and the potential ellipsoids.* This theorem is due to Sir Robert Ball.

7. If gravity be the only force acting on the body in the last example, show that the potential ellipsoid becomes a circular cylinder, and determine the positions of the harmonic axes.



Let O be the fixed point, G the initial position of the centre of inertia, G' its position resulting from a small angular displacement σ of the body round the line OR whose direction cosines are l, m, n. Draw GP and $G'P$ perpendicular to OR, and $G'H$ perpendicular to OG. Then $\sigma = \angle\, GPG'$, and the potential energy due to the displacement is $\mathfrak{M}g\,.\,GH$. Putting $OG = a$, and $\angle\, GOP = \rho$, we have

$$GH = \frac{GG'^2}{2a} = \frac{PG^2\,.\,\sigma^2}{2a} = \frac{a}{2}\sigma^2\sin^2\rho.$$

Again, if λ, μ, ν be the direction cosines of OG,

$$\sigma\cos\rho = \sigma(l\lambda + m\mu + n\nu) = \theta\lambda + \phi\mu + \psi\nu.$$

Hence, the potential energy due to the displacement is

$$\tfrac{1}{2}\mathfrak{M}ga\,\{\theta^2 + \phi^2 + \psi^2 - (\theta\lambda + \phi\mu + \psi\nu)^2\},$$

and the quadric E is determined by the equation

$$x^2 + y^2 + z^2 - (\lambda x + \mu y + \nu z)^2 = K,$$

which represents a right cylinder having OG for its axis.

This can also be easily seen directly as follows:—The amount of energy T required to turn the body through an angle σ round any semi-diameter r of E is, as we have seen, $\frac{1}{2}K\frac{\sigma^2}{r^2}$. Now, if r be vertical, $T = 0$, and therefore $r = \infty$. Also, if r be horizontal, $T = \mathfrak{M}gh$ (where h is the height through which the

centre of inertia is raised), and is therefore constant if σ be constant, hence r is constant: accordingly the corresponding section of E is a circle.

To determine the harmonic axes—One is the vertical, OG, the other two are found as follows:—Draw the diametral plane of the momental ellipsoid which is conjugate to a vertical line through the fixed point; it will meet the cylinder E and the momental ellipsoid in two ellipses; the pair of conjugate diameters common to these two are the lines required. Also, since a horizontal section of E is a circle, the projections on any horizontal plane of the two non-vertical harmonic axes are lines at right angles to each other. If the body be displaced without initial velocity there will be no oscillation round the vertical axis; and if the periods of vibration round the other two axes be different, the instantaneous axis of rotation will either oscillate or revolve continuously in the plane of the non-vertical harmonic axes. If after displacement an initial velocity be imparted to the body, in order that there should be small oscillations, this initial velocity of rotation must be round an axis in the plane of the two non-vertical harmonic axes.

8. If the system of forces acting on the body be constant in magnitude and direction, determine the values of q_{11}, &c. in Example 6.

Let X, Y, Z be the components of the constant force acting at any point of the body parallel to the initial direction of its principal axes, and let x, y, z be the coordinates of the point of application referred to space axes coinciding with these initial directions, its coordinates referred to the principal axes themselves being ξ, η, ζ. Then, if G_x, G_y, G_z be the moments of the forces round the space axes, neglecting small quantities of the second order, we have

$$G_x = \Sigma \{(yZ - zY) = \Sigma \{(\eta - \zeta\theta + \xi\psi) Z - (\zeta - \xi\phi + \eta\theta) Y\}$$

$$= \Sigma (\eta Z - \zeta Y) - \theta \Sigma (\eta Y + \zeta Z) + \phi \Sigma \xi Y + \psi \Sigma \xi Z$$

$$= \sigma \{- l \Sigma (\eta Y + \zeta Z) + m \Sigma \xi Y + n \Sigma \xi Z\},$$

since $$\Sigma (\eta Z - \zeta Y) = 0.$$

The work done by the forces in turning the body through $d\sigma$ is

$$(lG_x + mG_y + nG_z)\, d\sigma.$$

If we substitute for G_x its value given above, and make similar substitutions for G_y and G_z, we obtain, by integration, the terms of the second order in $-V$. Hence we have

$$q_{11} = \Sigma (\eta Y + \zeta Z), \quad 2q_{12} = - \Sigma (\xi Y + \eta X), \text{ \&c.}$$

9. If a system whose position is determined by three independent variables perform small oscillations, prove that the harmonic axes are the three conjugate diameters common to the quadrics whose equations are $\mathcal{T} =$ constant, $\mathfrak{S} =$ constant (Art. 313).

CHAPTER XIV.

THERMODYNAMICS.

321. **Mechanical Equivalent of Heat.**—The experiments of Joule and others have shown that whenever sensible kinetic energy disappears without a corresponding increase of potential energy, an amount of heat is produced proportional to the quantity of sensible kinetic energy which has disappeared.

A similar result takes place in all cases in which work is expended without producing a corresponding increase of energy; and, conversely, a definite amount of heat can be transformed into a definite amount of work.

The number of units of work which the unit of heat can perform is called the *mechanical equivalent of heat*, and may be designated by the letter J. If the quantity of heat required to raise the temperature of the unit mass of water from 0° to 1° Centigrade be taken as the *unit of heat*, and the amount of work expended in lifting the unit mass through a height of one metre as the unit of work, the value of J is *found to be* 424. In English units, *i. e.* if a foot be taken as the unit of length, and temperature be estimated by Fahrenheit's thermometric scale, the value of J is 772.

If Q be the number of units of heat imparted to a body; U its total energy, kinetic and potential; W the work done by the body against external forces; and ΔQ, &c. the increments of these quantities at any time reckoned from the same instant, the experiments of Joule, already mentioned, conduct to the equation

$$J\Delta Q = \Delta U + \Delta W. \tag{1}$$

If we desire to give this equation a purely theoretical basis, we have only to assume that heat is energy resulting from molecular motion, and that the principle of the conservation of energy holds good.

If we seek to determine J from Equation (1) by measuring the amount of ΔQ, &c. in any particular case, we are met by the difficulty that the value of ΔU is in general unknown. This difficulty can be got over by bringing back the body which is being experimented on to its initial condition: ΔU is then zero, and we have $J\Delta Q = \Delta W$.

When J is known, heat can be expressed in work units; and if this mode of expressing Q be adopted, (1) takes the simpler form

$$\Delta Q = \Delta U + \Delta W. \qquad (2)$$

322. **Equation of Energy.**—The equation of the preceding Article is one of the two fundamental equations of Thermodynamics, and may be called the Equation of Energy.

In the application of this equation the substance under consideration is in general supposed to pass continuously from one state to another, in consequence of changes in its temperature and in the pressure which the unit area of its surface exerts against the surrounding medium, this pressure being supposed the same at all points of the surface.

In Thermodynamics we are not usually concerned with the kinetic energy of sensible motion. In fact, the apparent effect of heat on a substance is either to raise its temperature, or to change its condition, or to cause it to do external work. Hence the body is, in general, supposed to be *at rest* in the ordinary sense, and its kinetic energy is exhibited in the form not of sensible motion, but of those molecular motions on which temperature depends. This being understood, we see that the total energy of a unit mass of the substance at any time is a function of two independent variables—the temperature t, and the pressure p.

As the volume v of the unit of mass depends on p and t, we may take as independent variables any two of the three quantities t, p, and v.

If the work done by the body against external forces be, as is usually the case, the work which it does in consequence of its expansion against the pressure on its surface, and if we consider merely the *unit of mass*, it is easily seen that

$dW = pdv$, and hence we have from (1) the equation

$$JdQ = \frac{dU}{dt}\,dt + \frac{dU}{dv}\,dv + pdv. \tag{3}$$

323. **Specific Heat.**—The number of units of heat which must be imparted to the unit of mass of a homogeneous substance to raise its temperature one degree is called its *specific heat*, and is equal to the limit of $\frac{\Delta Q}{\Delta t}$.

The specific heat in general depends on the external work accomplished by the body, and is indeterminate unless the relation between the variations of the independent variables be assigned. If, however, the temperature of the body be raised either under the condition that its volume remains constant, or under the condition that the pressure on its surface remains constant, the specific heat is a definite function of the variables on which the state of the body depends.

The *specific heat at constant volume* may be designated by C_v, and that at *constant pressure* by C_p. We have then, from (3),

$$JC_v = \left(\frac{dU}{dt}\right)_v, \quad JC_p = \left(\frac{dU}{dt}\right)_p + p\,\frac{dv}{dt}, \tag{4}$$

where $\left(\frac{dU}{dt}\right)_v$ indicates the differential coefficient of U with respect to t under the hypothesis that v is constant, and $\left(\frac{dU}{dt}\right)_p$ has a similar signification in reference to p, and where in the equation for C_p we regard v as a function of p and t.

For practical purposes, when great accuracy is not required, no distinction is made in the case of solid and liquid bodies between the two specific heats, and the specific heat for each body is assumed to be an absolute constant.

Examples.

1. A raindrop falls to the ground from a height of 1272 metres; determine by how much its temperature is raised, assuming that it imparts no heat to the air or to the ground. *Ans.* 3°C.

2. Find how much heat is disengaged if a bullet weighing 50 grammes and having a velocity of 50 metres per second strikes a target, assuming g to be 9·8 metres per second.

Ans. An amount of heat sufficient to raise one gramme of water through 15°C.

3. Supposing the Earth to have been originally a nebulous mass dissipated through space; find the heat produced by its condensation.

If $\mathcal{J}$ be the kinetic energy generated by the coming together of the nebulous mass, we have, by Ex. 12, Art. 138, $\mathcal{J} = \frac{3}{5}\frac{\mu m^2}{r}$. The equivalent amount of heat Q is given by the equation $Q = \frac{\mathcal{J}}{J} = \frac{3}{5}\frac{\mu m}{r^2}\frac{mr}{J} = \frac{3}{5}\frac{mgr}{J}$. Now, substituting for r its value in metres $\frac{20,000,000}{\pi}$, and 424 for J, we obtain $Q = 9000mg$, approximately. Hence the quantity of heat generated by the condensation of the Earth is 90 times the amount required to raise an equal mass of water from 0° to 100°C.

4. Find the amount of heat generated by the condensation of the sun.

Let Q' be the amount of heat required, then M and R being the mass and radius of the sun, we have, from Ex. 3, $\frac{Q'}{Q} = \frac{M^2}{m^2}.\frac{r}{R} = \frac{M}{m}.\frac{r}{R}.\frac{M}{m}$. Now, $\frac{M}{m} =$ 324000, and $\frac{R}{r} = 108$, approximately. Hence $\frac{Q'}{Q} = 3000\frac{M}{m}$, nearly. Again, $9000 \times 3000 = 27000000$, consequently the heat generated by the condensation of the sum is 270,000 times the amount required to raise the temperature of an equal mass of water from 0° to 100°C.

5. If the sum be contracting in consequence of its own attraction; determine the annual contraction which is required to maintain its temperature constant.

As in Ex. 3, we have $QR = \frac{3}{5}\frac{\mu M^2}{J}$, and therefore $\frac{\delta Q}{Q} + \frac{\delta R}{R} = 0$.

By observing the quantity of heat received from the sun in a given time by a given area on the surface of the Earth, it is easy to determine the whole amount of heat emitted by the sun in one year. From this, and the mass of the sun, we can ascertain that the temperature of an equal mass of water would be lowered a little more than 2° by losing this amount of heat. Consequently, $\frac{\delta Q}{Q} = \frac{1}{13000000}$, approximately, and therefore to maintain its present temperature the sun should contract each year by an amount sufficient to diminish its diameter by $\frac{1}{13000000}$ of its length.

324. **Perfect Gas.**—In the case of a perfect gas the volume v of the unit of mass is connected with the pressure p and the temperature t by an equation of the form

$$vp = v_0 p_0 (1 + at), \tag{5}$$

where v_0 is the volume corresponding to the pressure p_0 and the temperature zero, and a is a constant which is the same for all gases, its value being $\frac{1}{273}$ when temperatures are counted on the Centigrade thermometer. If the zero of temperature be taken at $-273°$ C., and temperature reckoned from this origin be denoted by T, we have, putting R for av_0p_0,

$$vp = RT. \tag{6}$$

The experiments of Joule and Thomson have shown that if the volume of a gas vary without any heat being imparted or abstracted, the temperature remains constant, provided no external work is done. If now in (2) we make $\Delta Q = 0$ and $\Delta W = 0$, we have $\Delta U = 0$; hence it appears, that if the temperature of a gas remains invariable so likewise does the internal energy, which is therefore a function of the temperature alone. In this case, by (4) and (6), we have

$$JC_p = JC_v + p\left(\frac{dv}{dt}\right)_p = JC_v + R. \tag{7}$$

The experiments of Regnault have shown that the specific heat of a gas at constant pressure is independent of the pressure, being a constant for each gas. From this it follows by (7) that the specific heat at constant volume is likewise a constant. In the case of a perfect gas equation (3) accordingly becomes

$$JdQ = JC_v dT + pdv. \tag{8}$$

If Q be the heat imparted to the unit mass, and c_v the specific heat at constant volume, expressed in work units, (8) may be written

$$dQ = c_v dT + pdv. \tag{9}$$

Again it is plain that

$$dU = c_v dT, \tag{10}$$

and, if c_p be the specific heat at constant pressure expressed in work units, that

$$c_p = c_v + R. \tag{11}$$

EXAMPLES.

1. Calculate the difference between the two specific heats of air, being given that a cubic metre of air at a temperature of 0° C. and under a pressure of 760 mm. of mercury, whose density is 13·6, weighs 1·2932 kilogrammes.

Ans. 0·069.

2. For any gas whose density referred to air is d, show that

$$C_p = C_v + \frac{0·069}{d}.$$

3. Determine the quantity of heat which must be imparted to a gas to enable it to expand at a constant pressure p_1 from the volume v_1 to the volume v_2.

Ans. $Q = \frac{C_p}{R} p_1(v_2 - v_1)$.

4. If T be the absolute temperature of a gas, and $\mathcal{T}'$ the portion of the energy of its unit mass which is due to the velocities of translation of its molecules, show that $\mathcal{T}' = \frac{3}{2}RT$.

Since $pv = \frac{2}{3}\mathcal{T}'$ (Ex. 18, Art. 288), this result follows from (6) Art. 324.

5. Determine the mean velocity of translation of a molecule of air which is at a temperature of 0° C.

Here T is 273, and the mean velocity required is 485 metres per second, nearly.

6. Show that the mean velocities of translation of the molecules of different gases when at the same temperature are inversely proportional to the square roots of the densities of the gases.

7. Determine the relation between the total kinetic energy $\mathcal{T}$ of a gas and that portion $\mathcal{T}'$ of the kinetic energy which is due to the velocities of translation of its molecules.

The total energy U of a unit mass of a gas is composed of the kinetic energy $\mathcal{T}$, and of the potential energy V, which again is the sum of two parts, V_1 resulting from the mutual action of the molecules, and V_2 depending on the constitution of the individual molecules. V_2 may be considered constant so long as the chemical constitution of the gas remains unchanged, and V_1 may be assumed to be zero, since U is a function of the temperature alone, and V_1, if it existed, would depend on the mutual distances of the molecules, and therefore on the volume. Hence $U = \mathcal{T} + V_2$.

Again, $\frac{dU}{dT} = JC_v =$ constant, whence $U = JC_vT + C'$, or $\mathcal{T} + V_2 = JC_vT + C'$.

Let $\mathcal{T} = \beta\mathcal{T}'$, then $\mathcal{T} = \frac{3}{2}\beta RT$, and $\frac{3}{2}\beta RT + V_2 = JC_vT + C'$. Hence, as V_2 is constant, β must be of the form $\gamma + \frac{\gamma'}{T}$, where γ and γ' are constants; and

$\mathcal{T}$ must be the sum of two parts—one proportional to the temperature, the other constant. The existence of the latter part seems in the highest degree improbable; we may, therefore, conclude that β is constant. To determine its value we have $\frac{2}{3}\beta R = JC_v$, whence $\beta = \frac{3}{2}\frac{1}{k-1}$ by (7), where $k = \frac{C_p}{C_v}$. Now k is found to be almost the same for all gases, and to be equal to 1·408; hence β is approximately the same for all gases, and is equal to 1·634.

8. Two masses of different gases have equal volumes at the same pressure and temperature; show that for all equal temperatures they have equal kinetic energies.

325. **Reversibility and Cyclical Processes.**—When a body experiences transformations such that the inverse changes can take place in precisely the same circumstances, the transformation is said to be *reversible*. In order that this should be the case, any source from which the body derives heat, or to which the body imparts heat, must, at the time at which the heat is transferred, be of the same temperature as the body; and also the external pressure on the body at any time must be equal to the pressure corresponding to the state of the body at the time.

A cyclical process is a transformation at the end of which the body returns to the same state as that in which it was at the beginning.

326. **Indicator Diagram.**—The state of a body is, as we have seen, a function of *two independent variables*. If those selected be the volume of the unit of mass and the pressure on the unit of area, the state of the body at any time is indicated by the position of a point whose coordinates referred to two rectangular axes are proportional to the volume and pressure.

In the case of a body undergoing a transformation according to a fixed law, the set of points indicating its successive states form a curve. In a reversible transformation, if no heat be lost or gained by the body during the transformation, this curve is called an *adiabatic* or *isentropic* curve. If the temperature remain constant the curve is called *isothermal*. The area comprised between the curve, its extreme ordinates, and the axis of abscissas, represents the work done by the body during the transformation.

327. Isothermals and Adiabatics for a Perfect Gas.—In the case of a perfect gas the isothermal curve is determined by the equation

$$pv = RT_1, \tag{12}$$

where T_1 is the constant temperature. The isothermal for a perfect gas is therefore an equilateral hyperbola.

Since the temperature remains constant the heat required to effect the transformation is given by making $dT = 0$ in (8), that is by the equation

$$JQ = \int_{v_1}^{v_2} pdv = RT_1 \int_{v_1}^{v_2} \frac{dv}{v} = RT_1 \log\left(\frac{v_2}{v_1}\right). \tag{13}$$

When a body undergoes an adiabatic transformation, $dQ = 0$, and therefore in this case (8) becomes

$$JC_v dT + pdv = 0. \tag{14}$$

If we substitute in this the values of dT and R derived from (6) and (7), and put $C_p = kC_v$, we get

$$kpdv + vdp = 0, \quad \text{that is} \quad k\frac{dv}{v} + \frac{dp}{p} = 0.$$

Integrating we have

$$pv^k = p_1 v_1{}^k, \tag{15}$$

where p_1 and v_1 are the initial pressure and volume.

The temperature T at any stage of an adiabatic transformation is given by the equation

$$\frac{T}{T_1} = \frac{pv}{p_1 v_1} = \left(\frac{v_1}{v}\right)^{k-1}. \tag{16}$$

Examples.

1. In an adiabatic transformation determine the equation connecting the initial and final pressures of a gas with its initial and final volumes.

Ans. $\dfrac{p_2}{p_1} = \left(\dfrac{v_1}{v_2}\right)^k.$

2. Determine the external work done by a gas in an isothermal transformation.

In this case $W = JQ = RT_1 \log \frac{v_2}{v_1} = p_1 v_1 \log \frac{v_2}{v_1}.$

3. Prove that the external work done by the unit mass of a gas in an adiabatic transformation is $JC_v(T_1 - T_2)$, where T_1 and T_2 are the initial and final temperatures.

4. If the decrease in the temperature of the air as its height above the surface of the earth increases were due merely to the fall of temperature resulting from the expansion caused by diminution of pressure, show that ΔT, the excess of the temperature at the earth's surface above the temperature at any height z, would be given by the equation

$$\Delta T = \frac{k-1}{k} \frac{273z}{h_0},$$

where h_0 is the height of a homogeneous atmosphere at 0° C.

If p be the pressure, and ρ the density of the air at the height z, we have, from the fundamental equation of hydrostatics, $dp = -g\rho\, dz$; but $\rho = \frac{1}{v} = \frac{p}{RT}$, and since the expansion is adiabatic, $\frac{dT}{T} = \frac{k-1}{k} \frac{dp}{p}$. Eliminating dp, we have $dT = -\frac{k-1}{k} \frac{g}{R} dz$; from this, since $gh_0 = p_0 v_0$, we obtain the equation given above.

328. **Fundamental Principles of Thermodynamics.**—The science of Thermodynamics is founded on two fundamental Principles. Of these, the first finds its mathematical expression in Equation (1), and involves two statements, viz. that *In every natural process the total energy is invariable;* and that *Heat is a form of energy, a definite amount of heat being equivalent to a definite amount of work.*

The second fundamental Principle was first stated by Clausius, as follows:—*It is impossible for a machine, unaided by external energy, to convey heat from one body to another at a higher temperature.*

By Thomson the same Principle is stated somewhat differently in the following manner:—*It is impossible by means of inanimate material agency to derive mechanical effect from any portion of matter by cooling it below the temperature of the coldest of the surrounding objects;* and by Clerk Maxwell in another form, thus:—*It is impossible, by the unaided action of natural processes, to transform any part of the heat of a body*

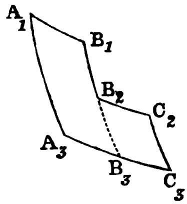
Hence, whatever be the body employed, we obtain the equation

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2}. \qquad (19)$$

331. **Extension of Carnot's Cycle.**—If heat imparted to a body be regarded as positive, and heat given out by the body as negative, (19) may be written

$$\frac{Q_1}{T_1} + \frac{Q_2}{T_2} = 0. \qquad (20)$$

If we now suppose a reversible cyclical process represented by any number of isothermals and adiabatics, each isothermal being followed by an adiabatic, and if Q be the number of units of heat imparted to the body at the temperature T, we have the equation $\Sigma \frac{Q}{T} = 0$.



In order to prove this, let us first suppose a cycle in which there are three isothermals, A_1B_1, B_2C_2, and A_3C_3, corresponding to the temperatures T_1, T_2, and T_3. Produce the adiabatic B_1B_2 to B_3, then $Q_3 = q_3 + q_3'$, where q_3 corresponds to B_3A_3, and q_3' to C_3B_3.

Now by (20),

$$\frac{Q_1}{T_1} + \frac{q_3}{T_3} = 0, \text{ and } \frac{Q_2}{T_2} + \frac{q_3'}{T_3} = 0;$$

from which, by addition, we have

$$\frac{Q_1}{T_1} + \frac{Q_2}{T_2} + \frac{Q_3}{T_3} = 0.$$

This result may be extended in a similar manner to a cycle containing four isothermals, and so on. Hence, in general,

$$\Sigma \frac{Q}{T} = 0. \qquad (21)$$

Again (21) holds good for every reversible cyclical process, whatever be the nature of the curves by which it is represented.

This appears from the consideration that two infinitely near points A and B on any curve can be connected by the element of an isothermal followed by that of an adiabatic, and that the area bounded by these elements, the ordinates of A and B, and the axis of abscissas, differs only by an infinitely small quantity of the second order from the area of which the arc AB is the boundary.

For every reversible cyclical process, however effected, we have, then, the equation

$$\int \frac{dQ}{T} = 0. \tag{22}$$

332. **Entropy.**—If a body pass from any one state to any other, we may suppose the change of state effected by means of a reversible transformation; and, whatever this process be, $\int \frac{dQ}{T}$ between the limits corresponding to the two states must have the same value, since the cycle may be completed by a definite invariable transformation. Hence $\int \frac{dQ}{T}$ depends only on the state of the body, and is independent of the mode (supposed reversible) by which the body is brought into this state.

If we put $\int \frac{dQ}{T} = \phi$, the quantity ϕ is called by Clausius the *Entropy* of the body.

The second fundamental Principle of Thermodynamics leads therefore to the result, that the entropy ϕ is a function of the two independent variables on which the state of the body depends, and therefore that in all *reversible transformations* $\frac{dQ}{T}$ is a perfect differential $d\phi$, or that

$$dQ = Td\phi. \tag{23}$$

In theoretical applications of the equation of entropy, Q and ϕ are supposed to be expressed in *mechanical* units.

333. **Energy and Entropy.**—For every reversible transformation in which the external work done by the body is due to its own expansion we have, if Q be expressed in work units, the two equations

$$\left.\begin{aligned} dQ &= dU + pdv \\ dQ &= Td\phi \end{aligned}\right\}. \qquad (24)$$

The energy U and the entropy ϕ are functions of the independent variables on which the state of the body depends, and dU and $d\phi$ are therefore *perfect differentials;* Q depends not merely on the state of the body but also on the mode in which it has been brought into that state; hence dQ is not a perfect differential. The limits of the quantities $\frac{\Delta Q}{\Delta v}$, $\frac{\Delta Q}{\Delta p}$, &c. are expressible in terms of the independent variables and the differential coefficients of U and v. They are therefore functions of the two independent variables which determine the state of the body, but are not differential coefficients. They may be written $\frac{\delta Q}{\delta v}$, &c.

Again from equations (24), we have

$$dU = Td\phi - pdv. \qquad (25)$$

In this equation if we select successively as independent variables ϕ, v; ϕ, p; T, p; T, v; and v, p; and express in each case the condition that dU should be a perfect differential, we obtain a system of equations which hold good in any reversible transformation in which the external work done by a body is due to its expansion against the pressure on its surface, and which are as follows:—

$$\left.\begin{aligned} &\left(\frac{dT}{dv}\right)_\phi = -\left(\frac{dp}{d\phi}\right)_v, \quad \left(\frac{dT}{dp}\right)_\phi = \left(\frac{dv}{d\phi}\right)_p, \quad \left(\frac{d\phi}{dp}\right)_T = -\left(\frac{dv}{dT}\right)_p \\ &\left(\frac{d\phi}{dv}\right)_T = \left(\frac{dp}{dT}\right)_v, \quad \left(\frac{dT}{dp}\right)_v\left(\frac{d\phi}{dv}\right)_p - \left(\frac{dT}{dv}\right)_p\left(\frac{d\phi}{dp}\right)_v = 1 \end{aligned}\right\}. \qquad (26)$$

Briot remarks that from the first of these equations the three succeeding can be obtained by interchanging p and v, or T and ϕ, the sign of the right-hand member of the equation being altered after each interchange.

334. **Elasticity and Expansion.**—The *elasticity* of a substance may be defined as the limit of the ratio of an increase of pressure to the compression which it produces, the compression being the ratio of the diminution of volume to the original volume.

The state of a substance being determined by two independent variables, except some connexion between their variations be assigned, the elasticity is indeterminate. The two elasticities usually considered are the elasticity at constant temperature E_T, and the elasticity at constant entropy E_ϕ. The former obviously belongs to an isothermal, and the latter to an adiabatic, transformation.

From the definitions of E_T and E_ϕ we have

$$E_T = -v\left(\frac{dp}{dv}\right)_T, \quad E_\phi = -v\left(\frac{dp}{dv}\right)_\phi. \tag{27}$$

When a body is heated it usually expands. The *expansion* is the ratio of the increase of volume to the original volume, and the *expansibility* is the limit of the ratio of the expansion to the increase of temperature, the pressure remaining constant. If e denote the expansibility, we have

$$e = \frac{1}{v}\left(\frac{dv}{dT}\right)_p. \tag{28}$$

If the expansion of a body take place without change of temperature, the limit of the ratio of the heat required for the expansion to the increase of volume is called the *latent heat of expansion*, and may be denoted by l; hence

$$l = \frac{1}{J}\left\{\left(\frac{dU}{dv}\right)_T + p\right\}. \tag{29}$$

The expansibility of a substance is called by some writers its *coefficient of dilatation.*

EXAMPLES.

1. The volume and pressure of a gas being given, determine its entropy.

$$\textit{Ans.}\ \phi - \phi_0 = J\left\{C_v \log\frac{p}{p_0} + C_p \log\frac{v}{v_0}\right\}.$$

2. Show that

$$JC_v = T\left(\frac{d\phi}{dT}\right)_v, \quad JC_p = T\left(\frac{d\phi}{dT}\right)_p, \quad J(C_p - C_v) = -T\frac{\left(\dfrac{dp}{dT}\right)_v^2}{\left(\dfrac{dp}{dv}\right)_T}.$$

The first two of these equations follow at once from (23). To prove the last, we have from the two former,

$$J(C_p - C_v) = T\left\{\left(\frac{d\phi}{dT}\right)_p - \left(\frac{d\phi}{dT}\right)_v\right\};$$

but

$$\left(\frac{d\phi}{dT}\right)_p = \left(\frac{d\phi}{dT}\right)_v + \left(\frac{d\phi}{dv}\right)_T\left(\frac{dv}{dT}\right)_p;$$

and by (26),

$$\left(\frac{d\phi}{dv}\right)_T = \left(\frac{dp}{dT}\right)_v; \text{ hence } J(C_p - C_v) = T\left(\frac{dp}{dT}\right)_v\left(\frac{dv}{dT}\right)_p.$$

Again, $dp = \left(\dfrac{dp}{dT}\right)_v dT + \left(\dfrac{dp}{dv}\right)_T dv$, from which, by making $dp = 0$, we get $\left(\dfrac{dv}{dT}\right)_p$, and substituting in the expression for $J(C_p - C_v)$ already given, we obtain the result required.

3. Prove that $\dfrac{C_p}{C_v} = \dfrac{E_\phi}{E_T}$.

$$\frac{C_p}{C_v} = \frac{\left(\dfrac{d\phi}{dT}\right)_p}{\left(\dfrac{d\phi}{dT}\right)_v} = \frac{\left(\dfrac{d\phi}{dv}\right)_p\left(\dfrac{dv}{dT}\right)_p}{\left(\dfrac{d\phi}{dp}\right)_v\left(\dfrac{dp}{dT}\right)_v}; \quad \text{but} \quad d\phi = \left(\frac{d\phi}{dv}\right)_p dv + \left(\frac{d\phi}{dp}\right)_v dp,$$

and therefore $\dfrac{\left(\dfrac{d\phi}{dv}\right)_p}{\left(\dfrac{d\phi}{dp}\right)_v} = -\left(\dfrac{dp}{dv}\right)_\phi$. In like manner $\dfrac{\left(\dfrac{dT}{dp}\right)_v}{\left(\dfrac{dT}{dv}\right)_p} = -\left(\dfrac{dv}{dp}\right)_T$;

hence

$$\frac{C_p}{C_v} = \frac{\left(\dfrac{dp}{dv}\right)_\phi}{\left(\dfrac{dp}{dv}\right)_T} = \frac{E_\phi}{E_T}\ (\text{Art. } 334).$$

4. Prove that $dQ = c_p dT - evTdp$.

$$dQ = \left(\frac{\delta Q}{\delta T}\right)_p dT + \left(\frac{\delta Q}{\delta p}\right)_T dp$$
$$= c_p dT + T\left(\frac{d\phi}{dp}\right)_T dp,$$

but by (26) we have $\left(\frac{d\phi}{dp}\right)_T = -\left(\frac{dv}{dT}\right)_p$, and hence by substitution we obtain from (28) the required result.

5. Assuming that the square of the velocity of the propagation of sound is proportional to the elasticity of the medium divided by its density, show that in a gas the velocity of sound varies as $\sqrt{kRT}$.

Since the compression of the air during the passage of a wave of sound is very sudden, the compression may be regarded as adiabatic. Hence the velocity of sound varies as $\sqrt{E_\phi v}$, but $E_\phi = kE_T$ (Ex. 3), and $E_T = p$, therefore, &c.

By means of the results obtained in this Example and in Ex. 1, Art. 324, if the velocity of sound be determined by experiment, C_p and C_v can be calculated. Conversely, if C_p be known by experiment, C_v can be found from the velocity of sound, and hence the value of J can be determined.

6. Show that bodies which expand by heating are heated by compression; those which contract by heating are cooled by compression; and, if the temperature be maintained constant, determine the rate at which heat is given out or absorbed according as the pressure is increased.

If Q be the heat required to keep the temperature constant, the rate of absorption is $\left(\frac{\delta Q}{\delta p}\right)_T$; but

$$\left(\frac{\delta Q}{\delta p}\right)_T = T\left(\frac{d\phi}{dp}\right)_T = -T\left(\frac{dv}{dT}\right)_p = -veT. \quad (\textit{See } (26) \text{ and } (28)).$$

Hence δQ is negative if e be positive, and conversely.

7. Prove that in water not far from its maximum density the rise of temperature produced by an increase of pressure is given approximately by the formula,

$$\Delta t = \frac{(t + 273)(t - 4)}{2950000}\Delta p,$$

where t is expressed in degrees centigrade, and p in atmospheres.

If v_0 be the volume of the unit mass of water at 4°, when the density is a maximum, the empirical formula $v = v_0\left(1 + \frac{(t-4)^2}{144000}\right)$ represents, according to Kopp and Tait, the results of numerous experiments. From this formula we have approximately $e = \frac{t-4}{72000}$.

Hence, assuming the pressure of the atmosphere to be 1033 grammes on the square centimetre, we obtain the required result.

8. If the internal energy of a body be a function of its temperature alone determine the relation which must exist between v, p, and T.

In this case (25) becomes $Td\phi = \frac{dU}{dT} dT + pdv$, whence

$$d\phi - \frac{dU}{dT} \frac{dT}{T} = \frac{p}{T} dv.$$

The left-hand side of this equation is a perfect differential, and therefore $p = Tf(v)$, which is the relation required.

9. If a body be such that its energy increases uniformly with the temperature when the volume is constant, and uniformly with the volume when the temperature is constant, and that its specific heat at constant pressure is constant, determine the equation connecting volume, pressure, and temperature.

Here we must have $dU = adT + bdv$, where a and b are constants. Hence from (25) we get $Td\phi = adT + (b + p)\,dv$, whence $b + p = Tf(v)$. If by means of this last equation we express dv in terms of dp and dT, we have

$$dQ = \left(a - \frac{f^2}{f'}\right) dT + \frac{f}{f'} dp. \quad \text{Now } c_p = \left(\frac{\delta Q}{\delta T}\right)_p,$$

and therefore, if n be the constant value of c_p, we obtain $a - \frac{f^2}{f'} = n$. From this we have $(a - n) \frac{df}{f^2} = dv$, and integrating we get $(v + C)\,f = (n - a)$, where C is the constant of integration. Hence we have as the required relation

$$(b + p)(v + C) = (n - a)\,T.$$

10. If the specific heats of a body at constant pressure and at constant volume be each constant, show that the energy is a linear function of the volume and absolute temperature.

Let $c_v = m$, $c_p = n$, then $\left(\frac{dU}{dT}\right)_v = m$, and therefore $U = mT + f(v)$.

$$\text{Also } c_p = \left(\frac{dU}{dT}\right)_p + p\left(\frac{dv}{dT}\right)_p, \text{ whence } n = m + (f' + p)\left(\frac{dv}{dT}\right)_p. \qquad (a)$$

Again, from (25) we have

$$Td\phi = mdT + (f' + p)\,dv, \quad \text{whence} \quad f' + p = TF(v). \qquad (b)$$

If we differentiate this equation, and eliminate from the equation so obtained and equations (*a*) and (*b*) the two quantities p and $\left(\frac{dv}{dT}\right)_p$, we get

$$(n - m)\,f'' = T\{F^2 + (n - m)\,F'\}.$$

This equation cannot be true for all values of the independent variables T and v except each side vanish separately, hence we have $f'' = 0$, and therefore $f(v) = C_1 v + C_2$. Consequently $U = mT + C_1 v + C_2$.

11. If the specific heat of a body at constant volume be constant, and the expansibility at any temperature be the inverse of the absolute temperature, determine the equation connecting volume, pressure, and temperature, and find the energy in terms of the temperature and volume.

Here, as in Ex. 10, $U = mT + f(v)$, and $f'(v) + p = TF(v)$. Also $\frac{1}{v}\left(\frac{dv}{dT}\right)_p = \frac{1}{T}$, whence $v = T\psi(p)$. Eliminating T we have $vF(v) = \psi(p)\{f'(v) + p\}$. Differentiating with respect to p we get

$$\psi'(p)f'(v) = -\{\psi(p) + p\psi'(p)\}.$$

This equation cannot be true in general except

$$f'(v) = -C, \text{ and } \psi(p) + p\psi'(p) = C\psi'(p),$$

where C is constant.

Hence we obtain $(C - p)v = KT$, and $U = mT - Cv + C'$, where K and C' are constants.

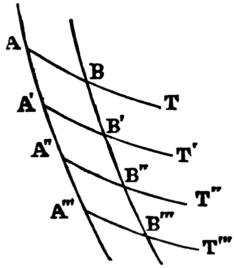
335. **Non-reversible Transformations.**—In the case of a non-reversible transformation we cannot assume the truth of equations (24). In fact, for such a transformation, even through the external work done by the body be due to its expansion against external pressure, this pressure need not be equal in magnitude to that belonging to the state of the body, nor is dQ in such a transformation necessarily equal to $Td\phi$.

In this case we must proceed as follows:—Let H be the heat actually imparted to the body in the non-reversible process, W the external work done, and U_0 and U the initial and final energies of the body; then

$$H = U - U_0 + W. \tag{30}$$

Let us now imagine a reversible transformation capable of bringing the body from its initial to its final state, but otherwise perfectly arbitrary. Since this hypothetical transformation is reversible, we can make use of equations (24) and (25), and of any results therefrom deducible to assist in determining $U - U_0$. The expression thus obtained may be substituted in (30).

the source, and W the heat converted into work in Carnot's cycle, $\frac{W}{Q}$ is a function of the extreme temperatures only, and is independent of the substance employed. In order, then, to construct a scale of temperature independent of any particular body we may proceed as follows:—



Draw the isothermal AB of a substance chosen at random, corresponding to any arbitrary temperature, which may be indicated by T, and draw the adiabatics AA' and BB' corresponding to the condition of the body before and after a certain arbitrary amount of heat Q has been imparted to it.

Draw another isothermal at a temperature T' less than T, so that the area $ABB'A'$ may be of given magnitude or correspond to a given amount of heat w. Now draw a series of isothermals T'', T''', &c., at intervals such that

$$ABB'A' = A'B'B''A'' = A''B''B'''A''' = \&c.;$$

then if $T - T'$ be the unit of temperature, $T - T''$ is two units, $T - T'''$ three units, &c.

Since T, Q, and w are fixed quantities, and W corresponding to $T^{(n)}$ is nw, Equation (18) shows that two bodies are at the same temperature if each indicates in the manner described n degrees of temperature below T. This method of estimating temperature is, therefore, independent of the body employed.

Again, if T' be *any* temperature lower than T estimated in this manner, and W' the heat converted into work in the corresponding cyclical process, we have $W' = (T - T')\,w$, and in like manner for another temperature T'' lower than T' we have $W'' = (T - T'')\,w$.

If we now suppose a cyclical process between the temperatures T' and T'', indicated by the points A', B', B'', A'', the heat converted into work is $W'' - W'$, and we get

$$W'' - W' = (T' - T'')\,w \qquad (31)$$

Again, the heat Q' drawn from the source at T', is equal to that given to the condenser in the process in which T and T' are the extreme temperatures; hence

$$Q - Q' = W' = (T - T')\, w, \text{ that is, } Q' = Q - (T - T')\, w. \quad (32)$$

337. **Efficiency of a Heat Engine.**—A system working in the manner required by Carnot's cycle may be termed a reversible heat engine, and the ratio of the heat converted into work to the heat drawn from the source is called the *efficiency of the engine.*

It appears by the reasoning of Art. 329 that the *extreme temperatures being given, the efficiency of a non-reversible engine cannot exceed that of a reversible, and that the efficiency of all reversible engines is the same.*

338. **Absolute Zero.**—From Art. 337 it appears that the efficiency of a reversible engine working between the temperatures T' and T'' is $\frac{W'' - W'}{Q'}$. By (31) and (32) this becomes $\dfrac{T' - T''}{T' - \left(T - \dfrac{Q}{w}\right)}$.

As T'' decreases, the efficiency increases, but the limit which it can never exceed is unity, since the mechanical work done by an engine can never exceed the equivalent of the heat drawn from the source. Hence, if we make the efficiency unity, we obtain for T'' the smallest possible value, which is $T - \frac{Q}{w}$. This temperature T'', since it is the lowest which can be attained by any body, must be the absolute zero. Hence

$$T - \frac{Q}{w} = 0, \quad \text{or} \quad \frac{w}{Q} = \frac{1}{T}.$$

The expression for the efficiency of a reversible engine working between any two temperatures T' and T'' becomes

then $\frac{T'-T''}{T'}$, and for the cyclical process described in Art. 329 we have $\frac{Q_1-Q_2}{Q_1}=\frac{T_1-T_2}{T_1}$. Carnot's function has thus been determined independently of the properties of any particular substance.

Again, this mode of determining Carnot's function shows that the existence of an absolute zero of temperature, suggested and rendered probable by the known properties of what are called permanent gases, follows necessarily from the two fundamental Principles of Thermodynamics.

The experiments of Joule and Thomson have shown that the absolute zero is 273·7 below zero on the Centigrade scale, or 460·66 below zero on the Fahrenheit. This is very nearly the same result as that of Article 324.

Examples.

1. The entropy ϕ being defined by the equation $dQ=f(t)\,d\phi$, prove that in a body subject to the equation $pv=Rf(t)$, where R is constant, the energy is a function of the temperature alone.

Let $f(t)=T$, then from (2) we have $d\phi=\frac{dU}{T}+R\,\frac{dv}{v}$. Hence $\frac{1}{T}dU$ must be a perfect differential, whence $U=F(T)$.

2. Gas is made to pass uniformly through a tube in which a porous plug, such as cotton-wool, is placed. No heat is permitted to leave the gas or enter it from any external source; determine the connexion between the variations of pressure and temperature caused by the plug.

Since the density of the gas at any particular cross section of the tube does not vary during the experiment, equal masses of gas pass through each section in the same time, or the velocity of the unit of mass is constant. Again, any energy which is lost by frictien is restored as heat. We are therefore entitled to assume that any change in the energy of the gas as it passes through different parts of the tube is due to the work done on it or to the work which it does.

Suppose two cross sections A and B of the tube, one on each side of the plug, the pressures at which are p_1 and p_2. As a small quantity dm of gas passes A the pressure driving it forward does work on it whose amount is p_1v_1dm. At the same time dm does work on the next layer of gas which is equal to the work done on dm when passing the section consecutive to A. Thus, in going from A to B the work done by dm and the work done on dm compensate each other, with the exception of p_1v_1dm done on dm, and p_2v_2dm done by dm. In other words, in the passage from A to B the whole external work done by dm is $(p_2v_2-p_1v_1)\,dm$, and therefore, since no heat is lost or gained, we have

$$U_2-U_1+p_2v_2-p_1v_1=0.$$

Now $U=\int Td\phi-\int pdv=\int Td\phi-pv+\int vdp$, or $U+pv=\int Td\phi+\int vdp$.

Hence the hypothetical reversible transformation (Art. 335) must be such that

$$\int\left\{T\left(\frac{d\phi}{dT}\right)_p dT+\left(T\left(\frac{d\phi}{dp}\right)_T+v\right)dp\right\}=0.$$

Substituting c_p for $T\left(\frac{d\phi}{dT}\right)_p$ (Ex. 2, Art. 334), and $-\left(\frac{dv}{dT}\right)_p$ for $\left(\frac{d\phi}{dp}\right)_T$ by (26),

we have $$\int\left\{c_p dT+\left(v-T\left(\frac{dv}{dT}\right)_p\right)dp\right\}=0.$$

Hence, if we can integrate the expression under the integral sign in this equation, the relation between T_2-T_1 and p_2-p_1 is determined. If the gas were theoretically perfect (Art. 324), we should have $T\left(\frac{dv}{dT}\right)_p=v$, and T_2 would be equal to T_1. This is found not to be the case. We may therefore conclude that no gas is theoretically perfect, and we cannot assume either that a is constant or that $T=t+\frac{1}{a}$.

From the equation $vp=v_0p_0(1+at)$, if we consider a as variable, we have

$$v-T\left(\frac{dv}{dt}\right)_p=\frac{p_0v_0}{p}\left(1+at-aT-tT\frac{da}{dt}\right).$$

We may assume that $a(T-t)$ differs from a constant by a very small quantity, and likewise that $\frac{da}{dt}$ is very small, and may integrate between T_1 and T_2 neglecting these quantities, since T_1-T_2 is observed to be small. For similar reasons we may assume that $\left(\frac{dv}{dT}\right)_p=\left(\frac{dv}{dt}\right)_p$, and that c_T is constant. Hence, integrating, we have

$$-c_p(T_2-T_1)=p_0v_0(1+at-aT)(\log p_2-\log p_1);$$

and therefore

$$T=t+\frac{1}{a}+\frac{c_p(T_1-T_2)}{ap_0v_0(\log p_1-\log p_2)}.$$

From this equation the exact position on the centigrade scale of the absolute zero can be determined (Art. 338).

339. **Change of State.**—There are three states or conditions in one of which matter is usually found; and which are termed the *solid*, the *liquid*, and the *gaseous*.

A solid body is one which strongly resists any forces tending to alter the relative positions of its adjacent molecules, and, so long as its structure is unbroken, admits of only slight changes in these positions.

A liquid offers scarcely any resistance to a change in the mutual position of its molecules, provided this change does not diminish the distance between those which are adjacent. In other words, it is indifferent to change of shape or separation of its molecules from each other, but strongly resists compression.

A gas, like a liquid, is indifferent to change of shape, but yields to compression with comparative facility, and tends to increase its volume without limit. To prevent its escape, it must therefore be restrained by an external envelope.

It is almost certain that every substance in nature is capable of existing in any one of the three states, and passes at a certain pressure and temperature from one of these states into another.

Thus, in general, there is a certain pressure and temperature, at which, if heat be imparted to a solid body and the pressure be maintained constant, the temperature does not rise, but the body gradually passes into the liquid state. The amount of heat required to bring about this change for the unit of mass is called the *latent heat of liquidity*. If the volume of the liquid exceed that of the solid, the latent heat of liquidity is spent partly in altering the internal energy and partly in doing external work. If (as is the case with water) the volume of the liquid be less than that of the solid, the internal energy of the liquid exceeds that of the solid by an amount greater than the equivalent of the latent heat of liquidity.

The vapour of a liquid may be considered a gas. If the temperature be sufficiently high, and the pressure sufficiently low, a vapour obeys the same laws as the gases which are called permanent, and approaches closely to the condition of a perfect gas (Art. 324).

For each vapour there is, corresponding to any given temperature T_1, a certain pressure p_1, such that at higher pressures the vapour begins to liquify; and, conversely, corresponding to any given pressure p_1 there is a temperature T_1, such that at lower temperatures the same result takes place. The pressure p_1 is called the maximum pressure of the vapour for the temperature T_1, and T_1 is called the boiling-point of the liquid for the pressure p_1.

In fact, if heat be imparted to the liquid under the constant pressure p_1, the temperature of the liquid will rise until it reaches T_1; after this the temperature will remain stationary, but the liquid will be transformed into vapour, and the heat required by the unit mass for this transformation is called the *latent heat of vaporization.* When a vapour is at its maximum pressure, and therefore beginning to liquify, it is said to be *saturated.*

A liquid exposed to the air evaporates more or less at all temperatures. It is known that if two gases be enclosed in the same envelope, each, after some time, is diffused through the whole volume of the envelope as if the other were absent. Hence we might anticipate the behaviour of a liquid exposed to the atmosphere, and expect that the air would not act like an impervious envelope exercising a constant pressure, but would merely retard the formation of vapour by diminishing its rate of diffusion.

The statements of Article 322 require some modification when applied to a body while changing its state. So long as the state remains unaltered, U, ϕ, and v are functions of p and T; but when the body begins to change its state, U, ϕ, and v vary, even though p and T remain constant. Whilst the body is changing its state, if p and T be constant, U, ϕ, and v are functions of a single variable. If the change of state go on continuously, and p at the same time vary, then T must also vary, and be at each instant a function of p. In this case, U and ϕ are functions of two independent variables, which may be v and p, or v and T, but cannot be p and T.

The experiments of Andrews, and the investigations of Thomson, have thrown much light on the phenomena of change of state, and enable us to explain their seeming anomalies. An account of these researches would, however, be outside the limits of the present work.

Examples.

1. If λ be the latent heat of change of state expressed in work units, and v the increase in the volume of the unit mass of the substance after passing from one state to the other, prove that

$$\lambda = vT\frac{dp}{dT}.$$

Let s and σ be the volumes of the unit mass of the substance before and after the change of state, and μ the fraction of the unit mass which has undergone the change at any instant during the transformation, then $v = \sigma - s$, and $v = \mu\sigma + (1 - \mu)s = s + v\mu$. Again, if Q be the quantity of heat required to transform μ times the unit mass from one state to the other, the pressure and temperature remaining constant, $Q = \lambda\mu$, and therefore

$$\lambda = \left(\frac{\delta Q}{\delta\mu}\right)_T = T\left(\frac{d\phi}{d\mu}\right)_T = vT\left(\frac{d\phi}{dv}\right)_T = vT\frac{dp}{dT}, \text{ by (26).}$$

The student will observe that p is here a function of T alone, and that if L be the latent heat expressed in heat units, $\lambda = JL$.

2. The density of ice being 0·92, and the latent heat of water 79·25, find the lowering of the temperature of freezing caused by an additional pressure of one atmosphere. *Ans.* 0·0073° C.

3. If c_s and c_w be the specific heats expressed in work units of saturated steam and of boiling water at the same pressure, show that

$$c_s - c_w = \frac{d\lambda}{dT} - \frac{\lambda}{T}.$$

It is here supposed that the variations of T and p are so related (Art. 339) that as T changes the steam remains saturated, and the water remains boiling. Hence, if we suppose μ to remain constant (Ex. 1), we have

$$c_s\mu + c_w(1 - \mu) = \left(\frac{\delta Q}{\delta T}\right)_\mu = T\left(\frac{d\phi}{dT}\right)_\mu;$$

and therefore

$$c_s - c_w = T\frac{d^2\phi}{dTd\mu} = T\frac{d}{dT}\left(\frac{d\phi}{d\mu}\right)_T;$$

but

$$\left(\frac{d\phi}{d\mu}\right)_T = \frac{\lambda}{T}, \text{ (Ex. 1)};$$

hence, substituting and performing the differentiation, we have the result required. It may be observed that c_w does not sensibly differ from the specific heat of water at constant pressure.

4. Investigate a numerical formula for the specific heat of saturated steam at any given temperature.

According to Regnault, the whole quantity of heat required to raise the temperature of the unit mass of water from 0° to t° C. and evaporate it at that temperature is $606\cdot5 + 0\cdot305\,t$. Hence we have the empirical formula

$$L + \int_0^t C_w dt = 606\cdot5 + 0\cdot305\,t,$$

where C_w is the specific heat of boiling water expressed in heat units. Differentiating, we have $\frac{dL}{dt} + C_w = 0\cdot305$. For high temperatures Regnault's empirical formula may be replaced by the simpler formula of Clausius, viz. $L = 607 - 0\cdot708\,t$. If we express $\frac{L}{T}$, by means of the latter we have (Ex. 3),

$$C_s = 0\cdot305 - \frac{607 - 0\cdot708\,t}{273 + t} = 1\cdot013 - \frac{800\cdot3}{273 + t}.$$

For temperatures near 0° C. we may take $C_w = 1$; we thus get

$$C_s = 1 - \frac{796\cdot2}{273 + t}.$$

From the expressions obtained for C_s, we may conclude that, except the temperature be enormously high, the specific heat of saturated steam is negative. Hence it follows, that if saturated steam be compressed, the temperature after compression will be higher than that corresponding to saturation at the new pressure; or, in other words, saturated steam suffers no condensation, but becomes super-heated by adiabatic compression. Conversely, if saturated steam be contained in a vessel impervious to heat, a diminution of pressure will cause partial condensation. These results were first obtained theoretically by Clausius and Rankine, who, independently, arrived at them almost simultaneously. They have since been confirmed experimentally by Hirn. It seems not unlikely that the connexion between rain and a change of atmospheric pressure depends partly on the property of steam mentioned above.

5. If U_1 be the energy of the unit mass of saturated steam at T_1, and U_0 that of the unit mass of boiling water at T_0, prove that

$$U_1 = U_0 + \int_{T_0}^{T_1} \left(c_w - p\,\frac{ds}{dT}\right) dT + \lambda_1 - p_1 v_1.$$

Let us suppose that the unit mass is brought from the state of boiling water at T_0 and p_0 to that of saturated steam at T_1 and p_1, and that this transformation is effected by first bringing the water without evaporation, but continually boiling, from p_0, T_0, to p_1, T_1; and then evaporating at T_1, p_1.

Take as variables T and μ (Ex. 1); then, since $dU = dQ - pdv$, we have

$$dU = \left\{\left(\frac{\delta Q}{\delta T}\right)_\mu - p\left(\frac{dv}{dT}\right)_\mu\right\} dT + \left\{\left(\frac{\delta Q}{\delta \mu}\right)_T - p\left(\frac{dv}{d\mu}\right)\right\}_T d\mu.$$

Again, on the above hypothesis, when T varies μ is zero, and when μ varies T is constant. In general (Ex. 3),

$$\left(\frac{\delta Q}{\delta T}\right)_\mu = (c_s - c_w)\mu + c_w,$$

and therefore when $\mu = 0$, $\left(\frac{\delta Q}{\delta T}\right)_\mu = c_w$.

Again,

$$\left(\frac{dv}{dT}\right)_\mu = \frac{ds}{dT}, \quad \left(\frac{\delta Q}{d\mu}\right)_T = \lambda, \quad \left(\frac{dv}{d\mu}\right)_T = v, \text{ (Ex. 1).}$$

Hence, substituting, we have

$$U_1 = U_0 + \int_{T_0}^{T_1} \left(c_w - p\frac{ds}{dT}\right) dT + (\lambda_1 - p_1 v_1)\int_0^1 d\mu.$$

6. Saturated steam in a vessel containing no water is allowed to escape into the air; determine the quantity of heat which must be imparted to the unit of mass in order that it should remain saturated.

Let W be the external work done by the unit mass of the steam in escaping, T_1 and T_2 its initial and final temperatures; then H being the heat required, we have

$$JH = U_2 - U_1 + W.$$

Now $\qquad W = p_2(\sigma_2 - \sigma_1) = p_2(v_2 - v_1) + p_2(s_2 - s_1)$, (Ex. 1),

where σ and s are the volumes of the unit mass of steam and of water,

and $\qquad U_2 - U_1 = \int_{T_1}^{T_2} \left(c_w - p\frac{ds}{dT}\right) dT + \lambda_2 - \lambda_1 - p_2 v_2 + p_1 v_1$, (Ex. 5).

Hence $\qquad JH = \int_{T_1}^{T_2} \left(c_w - p\frac{ds}{dT}\right) dT + \lambda_2 - \lambda_1 + v_1(p_1 - p_2) + p_2(s_2 - s_1).$

Since $(s_2 - s_1)$ and $\frac{ds}{dT}$ are small, we may neglect them, and thus obtain

$$JH = \int_{T_1}^{T_2} \left(c_w + \frac{d\lambda}{dT}\right) dT + v_1(p_1 - p_2).$$

Now $\qquad \frac{1}{J}\left(c_w + \frac{d\lambda}{dT}\right) = 0{\cdot}305$, (Ex. 4);

hence $\qquad H = -0{\cdot}305\,(T_1 - T_2) + (p_1 - p_2)\dfrac{L_1}{T_1\left(\frac{dp}{dT}\right)_1}$, (Ex. 1).

$\left(\frac{dp}{dT}\right)_1$ is the limit of the ratio of the change of the maximum pressure of the

vapour to the corresponding change of temperature, and is hence easily found from a table of the temperatures of the boiling-point at different pressures. H is in general positive; *i. e.* if no heat be imparted to the expanding steam, some of it will condense.

7. In the preceding example, if the vessel from which the steam is escaping contain boiling water, determine the quantity of heat which must be imparted to the unit mass of steam in order that it should remain saturated.

In this case, as in Ex. 2, Art. 338, the external work W done by the unit mass of the steam is

$$p_2\sigma_2 - p_1\sigma_1, \text{ or } p_2v_2 - p_1v_1 + p_2s_2 - p_1s_1.$$

Hence, as $U_2 - U_1$ is the same as in the last example, if we neglect the comparatively small terms involving s_1 and s_2, we have approximately

$$JH = \int_{T_1}^{T_2}\left(c_w + \frac{d\lambda}{dT}\right)dT,$$

and therefore $$H = -\,0{\cdot}305\,(T_1 - T_2).$$

In this case, if no heat be abstracted, or imparted, the steam after it escapes is super-heated. If $T_1 - T_2$ be large, or the steam originally at high pressure, the super-heating is considerable and more than sufficient to vaporize any particles of water which the steam carries with it mechanically. Hence we can explain the known phenomenon that high pressure steam after escaping into the air is dry and does not scald, whereas, by low pressure steam, severe scalds may be inflicted.

340. **Available Energy.**—The work which can be accomplished by a quantity of heat Q_1 depends on the temperature of the source from which it is derived. If this temperature be T_1, and the lowest temperature which can be obtained T_0, the work which can be accomplished by means of Q_1 cannot exceed $(T_1 - T_0)\dfrac{Q_1}{T_1}$ (Art. 338), where Q_1 is expressed in work units.

If Q_1 pass from a source at T_1 to a source at T_2, the available energy is diminished by the quantity $\left(\dfrac{T_0}{T_2} - \dfrac{T_0}{T_1}\right)Q_1$.

If Q_1 leave a source at T_1, and Q_2 in consequence enter a source at T_2, the loss of available energy is

$$(T_1 - T_0)\frac{Q_1}{T_1} - (T_2 - T_0)\frac{Q_2}{T_2}, \text{ or } Q_1 - Q_2 - T_0\left(\frac{Q_1}{T_1} - \frac{Q_2}{T_2}\right).$$

341. **Dissipation of Energy.**—If the transference of heat from a source at T_1 to a source at T_2 take place through the medium of a reversible engine undergoing a cyclical process, $\frac{Q_1}{T_1} - \frac{Q_2}{T_2}$ is zero, and the loss of available energy is $Q_1 - Q_2$, which is the same as the work done. Thus the uncompensated loss of available energy is zero.

In the case of an engine undergoing a non-reversible cyclical process, $Q_1 - Q_2$ cannot be greater, and is usually less, than $(T_1 - T_2)\frac{Q_1}{T_1}$ (Art. 337), or $\frac{Q_1}{T_1} - \frac{Q_2}{T_2}$ has a negative value which may be denoted by $-N$. In this case the uncompensated loss of available energy is $T_0 N$.

By a method similar to that employed in Art. 331 this result can be extended to every non-reversible cyclical process. In this case, if Q be the heat which enters the engine at the temperature T, the quantity $\Sigma\frac{Q}{T}$ is negative, and the uncompensated loss of available energy is $-T_0\Sigma\frac{Q}{T}$.

To prove this, we have only to substitute for the actual process A a process B in which the cycles corresponding to each pair of temperatures are completed by reversible transformations, each of which is accomplished first in one direction, then in the opposite. As these transformations are passed through in both directions, the value of $\Sigma\frac{Q}{T}$ and of the uncompensated loss of available energy is the same for A as for B; but $\Sigma\frac{Q}{T}$ for B is the sum of the values of $\Sigma\frac{Q}{T}$ corresponding to the small cycles, since the remaining part of B forms one reversible cycle. Hence we obtain the required results.

The uncompensated loss of available energy is called the *Dissipation of Energy.*

From the present and preceding Articles it appears that this dissipation takes place whenever heat passes without the performance of work from a body at a higher to a body at a lower temperature, and also, in general, in non-reversible

cyclical processes. A strictly reversible process cannot be realized in nature, since the absence of friction and the perfect equality of internal and external pressures and temperatures cannot be attained. Hence we may conclude, that in natural processes there is, in general, an incessant dissipation of energy.

There is one class of irreversible transformations in which, according to Mr. Parker (*Philosophical Magazine*, June, 1888), there is no dissipation of energy. Mr. Parker in the Article referred to defines an *equilibrium path* to be one at every point of which the system is in equilibrium. The path corresponding to a reversible transformation is always an equilibrium path, but an equilibrium path is not necessarily reversible. As a result of experiments on the solubility of various substances, Mr. Parker has been led to adopt the conclusion that in an irreversible equilibrium cycle there is no dissipation of energy.

It is to be observed that the theory of dissipation depends on the assumption of a certain temperature as the lowest which is available. If the lowest available temperature were absolute zero there would be no dissipation of energy.

342. **Increase of Entropy.**—If an element of heat dQ pass from a body A, whose temperature is T_1, to another body B at a lower temperature T_2, and if we suppose the volumes of A and B to remain constant, the entropy of A is diminished by $\frac{dQ}{T_1}$, and that of B increased by $\frac{dQ}{T_2}$, and as $T_1 > T_2$, the whole entropy of A and B is increased.

Again, in a cyclical process, if we suppose the source A and the condenser B to remain at constant volume, in which case their temperatures will of course vary, $\Sigma \frac{Q_1}{T_1}$ is the loss of entropy by A, and $\Sigma \frac{Q_2}{T_2}$ the gain of entropy by B. Hence the entropy of the whole system is increased by the quantity $\Sigma\left(\frac{Q_2}{T_2} - \frac{Q_1}{T_1}\right)$. In a reversible process this quantity is zero, but in a non-reversible process it has in general a positive value N.

We have supposed A and B to remain at constant volume; but if this be not the case, the results obtained still hold good, provided the transformation applied to each of these bodies is reversible when each body is considered alone. Under these circumstances the uncompensated loss of available energy in a non-reversible cyclical process is equal to the product of the limiting temperature and the increase of the entropy of the system.

Since, according to Mr. Parker, there is no dissipation of energy in an equilibrium cycle even though it be irreversible, in such a cycle the entropy of the whole system is constant. Again, it would appear that the definition of entropy in Art. 332 is unnecessarily restricted, and that entropy may be defined as $\int \frac{dQ}{T}$ along any equilibrium path.

It would seem that the result of Mr. Parker's experiments might have been anticipated. For, when a system undergoes a transformation corresponding to an equilibrium path, the irreversibility of the transformation for the whole system can result only from the way in which heat is communicated to or leaves the system, or on the mode in which it passes from one part of the system to another part. We may therefore suppose the system divided into portions for each of which taken separately a reversible path may be assigned coinciding with the actual equilibrium path. If Q_1, Q_2, &c. be the quantities of heat which at any stage of the transformation have passed into these portions, U_1, U_2, &c. their energies, v_1, v_2, &c. their volumes, p_1, p_2, &c. their pressures, T_1, T_2, &c. their temperatures, and ϕ_1, ϕ_2, &c. their entropies, we have $dQ_1 = dU_1 + p_1 dv_1 = T_1\, d\phi_1$, since the path coincides with a reversible path. In like manner

$$dQ_2 = dU_2 + p_2 dv_2 = T_2\, d\phi_2,\quad dQ_3 = dU_3 + p_3\, dv_3 = T_3 d\phi_3,\ \text{\&c.}$$

Now, since the whole system is in equilibrium,

$$T_1 = T_2 = T_3 = \text{\&c.} = T,\quad p_1 = p_2 = p_3 = \text{\&c.} = p.$$

Hence, if ϕ be the entropy of the entire system, and Q the quantity of heat imparted to it,

$$d\phi = d\phi_1 + d\phi_2 + \&c. = \frac{dQ_1 + dQ_2 + \&c.}{T} = \frac{dQ}{T},$$

and therefore so far as the relation between heat imparted and entropy is concerned, the whole transformation may be treated as if it were reversible.

We may conclude from what has been said, that natural processes have a tendency to increase entropy, or, as stated by Clausius, the entropy of the universe tends to become a maximum.

343. **Path of Least Heat.**—Let us suppose that a body, whose entropy is ϕ_1, passes from the state A to the state B in which its entropy is ϕ_0, less than ϕ_1. If Q be the heat given out by the body when at the temperature T, and if S denote the value of $\Sigma\,\frac{Q}{T}$ for the whole process, S cannot be less than $\phi_1 - \phi_0$. To prove this, first suppose the transformation reversible, then $S = \phi_1 - \phi_0$. Next suppose the transformation non-reversible, and let the cycle be completed by a reversible process which brings the body from B to A. The value of $\Sigma\,\frac{Q}{T}$ for the cycle is then $S - (\phi_1 - \phi_0)$, and this must be positive (Art. 340); hence $S > \phi_1 - \phi_0$.

Let us now consider by what path a body, whose temperature can never be less than T_0, should pass from the state A to the state B at T_0, ϕ_0, so that the heat given out in the passage should be a minimum, no heat being supplied to the body from any external source.

Let H be the heat given out; then for a non-reversible transformation, since $T > T_0$, and since any element of heat which enters the body at T must have previously passed out of it at a temperature higher than T, we must have $H > T_0 S > T_0(\phi_1 - \phi_0)$. For a reversible transformation $H = \int_{\phi_0}^{\phi_1} T d\phi$, which is least when $T = T_0$. The least value of H is therefore $T_0(\phi_1 - \phi_0)$. Hence the path consists of an adiabatic at the entropy ϕ_1 from T_1 to T_0, and an isothermal

at T_0 from ϕ_1 to ϕ_0. Since $U_1 - U_0 = W + H$, where W is the work done by the body during the transformation, when H is least W is greatest, and the maximum work which a body can perform under the circumstances supposed is

$$U_1 - U_0 - T_0(\phi_1 - \phi_0).$$

Examples.

1. Prove that the available energy of any system of bodies is

$$\Sigma m_1 \int_{T_0}^{T_1} c_1 \frac{T - T_0}{T} dT,$$

where T_1 is the initial temperature of m_1, and c_1 its specific heat at constant volume.

2. If the system in Ex. 1 be enclosed in an envelope impermeable by heat, show that T_0 is determined by the equation

$$\Sigma m_1 \int_{T_0}^{T_1} c_1 \frac{dT}{T} = 0.$$

The actual work performed by the system during the transformation in which all its parts are brought to the temperature T_0 is

$$\Sigma m_1 \int_{T_0}^{T_1} c_1 dT;$$

but, if the transformation be that in which the greatest possible work is done, this work must be equal to the available energy, and therefore

$$\Sigma m_1 \int_{T_0}^{T_1} c_1 \frac{dT}{T} = 0.$$

When the limiting temperature T_0 is determined from within, as in this example, or, in other words, when one part of the system acts as condenser to another part, the available energy is called by Thomson the *Internal Thermodynamic Motivity.* When T_0 is independent of the system, *i.e.* when heat can pass out of the system to an external condenser, the available energy may be termed the *External Thermodynamic Motivity.* In this case T_0 must be assigned.

3. If a system consist of two equal masses of the same substance whose specific heat is constant, show that the limiting temperature of the internal thermodynamic motivity is $\sqrt{T_1 T_2}$, where T_1 and T_2 are the initial temperatures of the two masses.

4. In the preceding example prove that the thermodynamic motivity of the system is $mc(\sqrt{T_1} - \sqrt{T_2})^2$.

5. If the entropy of a substance be increased, its energy remaining constant, prove that the work which can be obtained by a transformation to a given state is diminished.

6. A unit mass of gas, whose volume is v_1, is allowed to expand into a perfectly empty vessel, whereby its volume becomes v_2; show that its capability of doing work is diminished by the quantity $T_0 R \log \frac{v_2}{v_1}$.

7. Determine a transformation by which, without the transference of any heat, gas at $p_1 v_1$ may be brought by the application of the smallest possible amount of external work to $p_2 v_2$; where $p_2 > p_1$, $v_2 > v_1$.

Since $v_2 > v_1$ the gas must expand, and since no heat is given it must expand by its own energy. It will do this with the smallest possible expenditure of energy by expanding into a vacuum. If U_2 be the energy corresponding to $p_2 v_2$, the smallest amount of external work capable of changing the energy from U_1 to U_2 is $U_2 - U_1$, and in order that no more than this should be required the compression must, by (25), be adiabatic. Hence let the gas expand into a vacuum till its volume become v, and then let it be compressed adiabatically till its volume become v_2. In order to determine v, let T_1 and T_2 be the temperatures belonging to the initial and final state; then, by (16),

$$T_1 v^{k-1} = T_2 v_2^{k-1}, \quad \text{whence} \quad v = v_2 \left(\frac{T_2}{T_1}\right)^{\frac{1}{k-1}}.$$

Miscellaneous Examples.

1. If two points fixed in a lamina slide upon two intersecting straight lines, and if one point be made to oscillate backwards and forwards so as to have always the same velocity, the ellipse described by any fixed point of the lamina will be described under acceleration which is fixed in direction.

2. A material point of given mass moves freely under the action of a central force of given absolute intensity, varying inversely as the square of the distance; given the initial circumstances of projection, determine the major axis, eccentricity, and line of apsides of the orbit it describes.

3. The extremities of a uniform rectilinear bar move on the circumference of a smooth vertical circle; find its period of oscillation under the action of gravity consequent on a small displacement from its position of stable equilibrium.

4. A circular plate, revolving round its centre in a vertical plane, becomes suddenly attached at its lowest point to a heavy particle previously at rest; required the mass of the particle in order that, at the end of a semi-revolution, the system may be brought to rest under the action of gravity.

5. A uniform beam is supported symmetrically on two props; find where they should be placed in order that if one of them be removed the instantaneous pressure on the other may be the same as the statical pressure.

6. A circular board lies upon a smooth table; in the board is cut a circular groove along which a molecule is projected with a given velocity; determine the pressure against the side of the groove.

7. A straight rod which passes through a small fixed ring is in motion in a horizontal plane; determine the motion of its centre of gravity.

8. A lamina unacted on by any force is projected in its own plane; prove that its space centrode is a straight line, and its body centrode a circle.

9. A sphere, rotating about a horizontal axis through its centre of gravity, falls vertically; prove that its space centrode is a parabola, and its body centrode a spiral of Archimedes.

10. Given the motion of one point in a body and also its space centrode, find its body centrode.

11. A small ring slides down a rough rod from a given point to a given right line; find the direction of the rod so that the time of descent may be a minimum.

(a) Find the limits of the coefficient of friction for which the required position is vertical.

12. A material particle, attached to a fixed point by an inelastic string, is allowed to descend a smooth inclined inelastic plane, starting without initial velocity from the foot of the perpendicular from the fixed point on the plane. Describe the subsequent motion, and show that the total length of the path described by the particle on the plane before it comes to rest is

$$l\left(\frac{1}{\sin\beta} - \frac{\sin\beta\cos^2\beta}{1+\cos^2\beta}\right),$$

where l is the length of the string, and β is the angle which, when stretched, it makes with the perpendicular.

13. A homogeneous sphere rolls down the concave surface of a rough semicircle, the axis of which is vertical; find its velocity and entire pressure against the semicircle in any position.

14. Two balls of different masses, moving in the same right line with different velocities, become suddenly connected by a weightless inextensible rod; given all particulars, required, in magnitude and direction, the initial strain on the rod.

15. A material particle, constrained to oscillate without friction in a curve tautochronous with respect to any point under the action of any force, being supposed retarded throughout its motion by a resistance to its velocity of constant intensity; determine the law of diminution of its several successive arcs of vibration.

16. The resisting, in the preceding, being supposed small compared with the moving force; show that, if the friction vary as any function of the velocity, its effect will be ultimately inappreciable on the time of description of any complete arc of vibration of the particle.

17. A rigid body, revolving round a fixed axis, strikes perpendicularly against a fixed obstacle; required the height through which the same body should fall vertically, without rotation, so as to strike against the obstacle with the same force of percussion.

18. A rigid body connected with a fixed point by an inextensible cord, is in constrained equilibrium under the action of a force passing through its centre of inertia; all the other restraints being supposed suddenly removed, required the initial stress on the cord.

19. A sphere, rolling without sliding on a rough horizontal plane, is acted on by a central force, varying inversely as the square of the distance, emanating from a fixed point in the parallel plane passing through its centre. Show that it describes a focal conic round the centre of force; and determine the initial velocity for which the motion is parabolic.

20. A rigid body, being set in motion by a single impulsive force, show that all axes of initial pure rotation, corresponding to different directions of the percussion, envelope a quadric cone, diverging from the centre of inertia, and touching the three central principal planes of the body.

21. A rigid body, having two fixed points, is set in motion by an impulsive force; determine in magnitude and direction the initial percussions at the points perpendicular to their line of connexion.

22. Two material particles, resting on a rough inclined plane, and connected by a slight flexible cord, passing without friction through a small ring attached to a fixed point on the plane, are in equilibrium under the action of gravity; the inclination of the plane being supposed gradually increased, or its roughness ness gradually diminished, determine the nature of the initial motion of the particles.

23. Two material particles, moving without friction in two non-intersecting rectilinear tubes of indefinite length, attract each other with a force varying directly as their distance asunder; determine completely their motion.

24. In the general displacement of a solid from one given position to another, find, by geometrical construction, the twist by which the body can be brought from the former to the latter position.—(Prof. Crofton, *London Mathematical Society*, 1874.)

Let A be any point of the solid in its first position, B the new position of the same point; again, let C be the new position of the point which was originally at B, and D the new position of that point originally at C; then, to find the required twist, bisect the angles ABC and BCD by the lines BH and CK; find HK the shortest distance between these bisectors. The body can be brought from the first to the second position by a translation HK, and a rotation round HK through an angle which is equal to that between BH and CK.

25. Calculate, in C. G. S. units, the mutual attraction of two units of mass at the unit distance apart, according to the law of gravitation.

Let γ denote the quantity in question; then the attraction of the earth on a unit of mass at its surface is $\frac{4}{3}\pi\gamma\rho R$, where ρ is earth's mean density, and R is its radius.

Hence we have $g = \frac{4}{3}\pi\rho\gamma R$.

Now, in the system of units adopted, we have $g = 981$, and $\pi R = 2 \times 10^9$. Thence, assuming $\rho = 5{\cdot}67$, we get

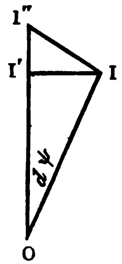
$$\frac{1}{\gamma} = \frac{8}{3} \times \frac{567}{981} \times 10^7 = \frac{168}{109} \times 10^7 = 15{,}410{,}000, \text{ approximately};$$

$$\therefore \quad \gamma = \frac{1}{15{,}410{,}000} \text{ dynes.}$$

26. A body is rotating about a fixed point. Express the element of the curve described by the instantaneous axis on a sphere fixed in the body in terms of the angular velocities round the body-axes.

Let the instantaneous axis at any time make angles λ, μ, ν with the body-axes; let the spherical surface be intersected by the two consecutive positions of the instantaneous axis in I and I'; let OI and OI'' represent the corresponding magnitudes ω and $\omega + d\omega$ of the angular velocity. Then the projections of II'' on the body-axes are proportional to $d\omega_1$, $d\omega_2$, $d\omega_3$, and $I'I''$ is proportional to $d\omega$.

Now $$II''^2 = I'I''^2 + OI^2 d\psi^2;$$



hence $$d\omega_1^2 + d\omega_2^2 + d\omega_3^2 = d\omega^2 + \omega^2 d\psi^2,$$

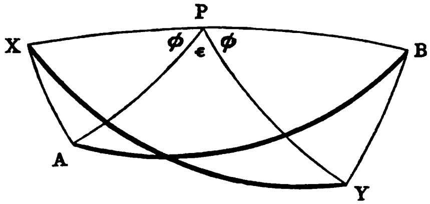
and $$d\psi^2 = \frac{1}{\omega^2}(d\omega_1^2 + d\omega_2^2 + d\omega_3^2 - d\omega^2).$$

If a be the radius of the sphere, and σ the arc of the curve, we have, therefore,

$$\left(\frac{d\sigma}{dt}\right)^2 = a^2\left(\frac{d\psi}{dt}\right)^2 = \frac{a^2}{\omega^2}\left\{\left(\frac{d\omega_1}{dt}\right)^2 + \left(\frac{d\omega_2}{dt}\right)^2 + \left(\frac{d\omega_3}{dt}\right)^2 - \left(\frac{d\omega}{dt}\right)^2\right\}.$$

27. A body is moving round a fixed point. Being given the axis, a rotation round which brings the body from one position to another, and the magnitude of the rotation, determine the angles which body-axes make in the second position with the space-axes which in the first position coincide with them.

Describe a sphere round the fixed point O. Let two of the space-axes meet



this sphere in the points X, Y; and the corresponding body-axes in the points A, B, when the body is in its second position; let P be the pole of rotation; then $XPA = YPB = \phi$, where ϕ is the given rotation. Let l, m, n be the direction cosines of the angles that OP makes with OX, OY, OZ.

Then $\cos XA = l^2 + (1 - l^2)\cos\phi$, and it can be readily shown that

$$\cos YA = lm(1 - \cos\phi) + n\sin\phi,$$

and $$\cos XB = lm(1 - \cos\phi) - n\sin\phi.$$

The values of the cosines of the remaining angles can now be written down from symmetry.

If we put

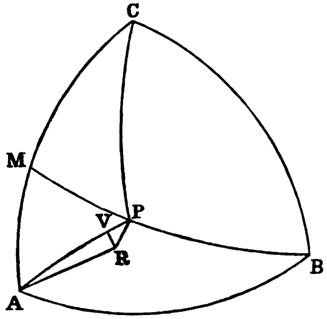
$$v = \cos\tfrac{1}{2}\phi, \quad \lambda = l\sin\tfrac{1}{2}\phi, \quad \mu = m\sin\tfrac{1}{2}\phi, \quad \nu = n\sin\tfrac{1}{2}\phi,$$

we have the following table of the values of $\cos XA$, &c. :—

	X	Y	Z
A	$v^2 + \lambda^2 - \mu^2 - \nu^2$	$2(\mu\lambda + v\nu)$	$2(\nu\lambda - v\mu)$
B	$2(\lambda\mu - v\nu)$	$v^2 + \mu^2 - \nu^2 - \lambda^2$	$2(\nu\mu + v\lambda)$
C	$2(\lambda\nu + v\mu)$	$2(\mu\nu - v\lambda)$	$v^2 + \nu^2 - \lambda^2 - \mu^2$

The quantities l, m, n, ϕ are called Rodrigues' coordinates. (Thomson and Tait, *Natural Philosophy*, § 95.)

28. The body is rotating round OA with an angular velocity ω_1; determine the differential coefficients of ϕ, l, m, n with respect to the time.



We have to find the magnitude and axis of the rotation which is the resultant of the rotation ϕ round the axis (l, m, n), and of $\omega_1 dt$ round OA. If this rotation be ϕ', and its direction cosines l', m', n'; $\phi' - \phi = \frac{d\phi}{dt}dt$, $l' - l = \frac{dl}{dt}dt$, &c.

Let A, B, C be the points in which OA, OB, OC meet the sphere described round O, and P the point in which the sphere is met by the line (l, m, n). As in Ex. 8, Art. 260, make $APR = \frac{1}{2}\phi$, and $PAR = -\frac{1}{2}\omega_1 dt$, then R is the pole of the resultant rotation; the positive direction of rotation being supposed to be counter-clockwise.

Draw RV at right angles to AP. Then it is easily seen that

$$\cos BPA = -\frac{lm}{\sqrt{\{(1-l^2)(1-m^2)\}}}, \quad \sin BPA = \frac{n}{\sqrt{\{(1-l^2)(1-m^2)\}}};$$

also
$$2\frac{dl}{dt} = \omega_1 \sin^2 AP \frac{\cos APR}{\sin\frac{1}{2}\phi} = \omega_1(1-l^2)\cot\tfrac{1}{2}\phi,$$

$$2\frac{dm}{dt} = \omega_1 \sin AP \sin BP \frac{\cos BPR}{\sin\frac{1}{2}\phi} = \omega_1(n - lm\cot\tfrac{1}{2}\phi),$$

$$2\frac{dn}{dt} = \omega_1 \sin AP \sin CP \frac{\cos CPR}{\sin\frac{1}{2}\phi} = -\omega_1(m + nl\cot\tfrac{1}{2}\phi).$$

Again
$$\tfrac{1}{2}\phi' = \pi - ARP,$$

and
$$\cos ARP = \frac{\omega_1 dt}{2}\cos AP \sin\frac{\phi}{2} - \cos\frac{\phi}{2};$$

hence
$$\cos\tfrac{1}{2}\phi' = \cos\tfrac{1}{2}\phi - \tfrac{1}{2}l\omega_1 \sin\tfrac{1}{2}\phi\, dt; \quad \therefore \frac{d\phi}{dt} = l\omega_1.$$

29. A body is moving round a fixed point O, with angular velocities $\omega_1, \omega_2, \omega_3$, round three rectangular axes OA, OB, OC fixed in the body. Determine the differential coefficients of Rodrigues' coordinates with respect to the time.

By means of the last example we can write down the changes produced on ϕ, l, m, n by each of the rotations $\omega_1 dt$, $\omega_2 dt$, $\omega_3 dt$.

Adding, and dividing by dt, we get

$$2\frac{dl}{dt} = -\omega_2 n + \omega_3 m + \cot\tfrac{1}{2}\phi\{\omega_1 - l(l\omega_1 + m\omega_2 + n\omega_3)\},$$

$$2\frac{dm}{dt} = -\omega_3 l + \omega_1 n + \cot\tfrac{1}{2}\phi\{\omega_2 - m(l\omega_1 + m\omega_2 + n\omega_3)\},$$

$$2\frac{dn}{dt} = -\omega_1 m + \omega_2 l + \cot\tfrac{1}{2}\phi\{\omega_3 - n(l\omega_1 + m\omega_2 + n\omega_3)\},$$

$$\frac{d\phi}{dt} = l\omega_1 + m\omega_2 + n\omega_3;$$

whence, also, we obtain

$$2\frac{dv}{dt} = -\omega_1\lambda - \omega_2\mu - \omega_3\nu, \quad 2\frac{d\lambda}{dt} = \omega_1 v - \omega_2\nu + \omega_3\mu,$$

$$2\frac{d\mu}{dt} = \omega_2 v - \omega_3\lambda + \omega_1\nu, \quad 2\frac{d\nu}{dt} = \omega_3 v - \omega_1\mu + \omega_2\lambda,$$

where v, λ, μ, ν have the same meaning as before.

30. A rigid body is moving in any manner; one point is suddenly arrested; determine the impulse exerted on the body.

Let u, v, w be the components of the velocity of the point immediately before it is arrested, x, y, z its coordinates, and X, Y, Z the components of the impulse, the axes being the principal axes of the body at the centre of inertia, then X is given by the equation

$$-\left\{\frac{ABC}{\mathfrak{M}} + A(B+C)x^2 + B(C+A)y^2 + C(A+B)z^2 + \mathfrak{M}Ir^4\right\}X$$
$$= \{ABC + \mathfrak{M}[A(B+C)x^2 + BC(y^2+z^2) + \mathfrak{M}Ir^2x^2]\}u$$
$$+ \mathfrak{M}(AB + \mathfrak{M}Ir^2)xyv + \mathfrak{M}(AC + \mathfrak{M}Ir^2)xzw,$$

where I is the moment of inertia of the body round the line joining the arrested point to the centre of inertia, r the distance between these points, and A, B, C, the principal moments of inertia of the body.

31. A sphere is projected in any way along an imperfectly rough inclined plane. Investigate the motion.

(This investigation, with some slight modifications, is taken from Routh, *Rigid Dynamics*.)

Here the equations of motion are

$$M\ddot{x} = X + Mg \sin i, \quad M\ddot{y} = Y,$$

$$\tfrac{2}{5}Mr^2\dot{\omega}_1 = rY, \quad \tfrac{2}{5}Mr^2\dot{\omega}_2 = -rX,$$

whence, eliminating X and Y, we obtain, on integrating,

$$\dot{x} + \tfrac{2}{5}r\omega_2 = gt \sin i + \alpha + \tfrac{2}{5}r\Omega_2,$$

$$\dot{y} - \tfrac{2}{5}r\omega_1 = \beta - \tfrac{2}{5}r\Omega_1,$$

where α, β, Ω_1, and Ω_2 are the initial values of $\dot{x}$, $\dot{y}$, ω_1, and ω_2.

Again, if u be the velocity at any instant of that point of the sphere which is in contact with the plane, and θ the angle which its direction makes with the axis of x,

$$u \cos\theta = \dot{x} - r\omega_2, \quad u \sin\theta = \dot{y} + r\omega_1.$$

Differentiating, substituting for $\ddot{x}$, &c., from the equations of motion, putting for X and Y the values which they take as long as there is slipping, viz., $-\mu Mg \cos i \cos\theta$ and $-\mu Mg \cos i \sin\theta$, and solving the resulting equations for $\dot{u}$ and $u\dot{\theta}$, we have

$$\dot{u} = g \sin i \cos\theta - \tfrac{7}{2}\mu g \cos i, \quad u\dot{\theta} = -g \sin i \sin\theta.$$

Hence, if $\frac{7}{2}\mu \cot i = n$, we obtain, by integration, $u \sin\theta = K_1 (\tan\frac{1}{2}\theta)^n$. Substituting the value given by this equation for u in the equation for $u\dot{\theta}$, and integrating, we have

$$\frac{(\tan\frac{1}{2}\theta)^{n+1}}{n+1} + \frac{(\tan\frac{1}{2}\theta)^{n-1}}{n-1} = K_2 - \frac{2g \sin i}{K_2}\, t.$$

K_2 is determined from the initial value of θ, and K_1 from the initial values of θ and u. These latter are given by the equations

$$u_0 \cos\theta_0 = \alpha - r\Omega_2, \quad u_0 \sin\theta_0 = \beta + r\Omega_1;$$

then u and θ being known, $\dot{x}$, $\dot{y}$, ω_1, and ω_2 can be determined.

If n or $\frac{7}{2}\mu \cot i > 1$, u and θ become continually less until they vanish together. Pure rolling then begins at a time t_0, which is given by the equation $t_0 = \dfrac{K_1 K_2}{2g \sin i}$. After pure rolling begins the values of $\dot{x}$, $\dot{y}$, ω_1, and ω_2, at any time, can be obtained from the combination of the equations of motion with the equations

$$\dot{x} - r\omega_2 = 0, \quad \dot{y} + r\omega_1 = 0.$$

If $n < 1$, θ, though constantly approaching zero, as appears from the expression for $u\dot{\theta}$, will not vanish in any finite time, and u tends to increase without limit.

If $u_0 = 0$, the problem is at starting reduced to that of Ex. 3, Art. 278. The force of friction requisite for pure rolling is then $\frac{2}{7} Mg \sin i$. Hence, if

$$\tfrac{2}{7} Mg \sin i < \mu' Mg \cos i, \quad \text{or} \quad \tfrac{7}{2} \mu' \cot i > 1,$$

where μ' is the coefficient of *statical* friction, pure rolling will commence and continue. If $\frac{7}{2} \mu' \cot i < 1$, slipping will begin at once and never cease.

32. A body rests with a plane face on an imperfectly rough horizontal plane. The centre of inertia of the body is vertically over the centre of inertia of the face and very near it, the connecting line being a principal axis at the former point. The form of the face is such, that its radii of gyration about all lines in it passing through its centre of inertia are equal. The body is projected with an initial velocity of translation U, and an initial very small angular velocity Ω round a vertical axis through its centre of inertia: determine the motion.

Take the initial direction of translation, and a horizontal line at right angles thereto for axes of x and y. Let u and v be the components of the velocity of the centre of inertia of the body at any time, and ω the angular velocity. Then, x and y being the coordinates of any point of the body, and ξ and η its coordinates referred to parallel axes through the centre of inertia,

$$\frac{dx}{dt} = u - \eta\omega, \quad \frac{dy}{dt} = v + \xi\omega.$$

If F be the magnitude of the whole force of friction at any point, its components X and Y are given by the equations

$$X = -F \frac{u - \eta\omega}{\sqrt{\{(u - \eta\omega)^2 + (v + \xi\omega)^2\}}} = -F, \qquad q.p.$$

$$Y = -F \frac{v + \xi\omega}{\sqrt{\{(u + \eta\omega)^2 + (v + \xi\omega)^2\}}} = -F \frac{v + \xi\omega}{u}, \qquad q.p.$$

since v, $\xi\omega$, and $\eta\omega$ are small compared with u.

Again, if S be the area of the plane face, the magnitude of the normal reaction of the horizontal plane on an element of the face is equal to $\phi(\xi, \eta)\, dS$, whence $F = \mu\phi(\xi, \eta)\, dS$, and, since $\bar{z} = \text{constant}$, $\int \phi(\xi, \eta)\, dS = mg$, where m is the mass of the body. Also equations (17), of Art. 267, give $G_x = 0$, $G_y = 0$, since $\omega_x = 0$, $\omega_y = 0$, $i = 0$, $j = 0$.

If a be the distance of the centre of inertia of the body from the plane face, and $\phi(\xi, \eta) = R$,

$$G_y = \mu a \int R dS - \int R\xi dS;$$

therefore $$\int R\xi dS = \mu mga.$$

Assume $R = K + \epsilon\Lambda$, where K and ϵ are constants, then

$$\mu mga = K \int \xi dS + \epsilon \int \Lambda\xi ds, \quad \text{but} \quad \int \xi dS = 0;$$

therefore ϵ must be small; also

$$mg = KS + \epsilon \int \Lambda dS.$$

Again, $$G_x = \int R\eta dS - \mu a \int \frac{v + \xi\omega}{u} RdS;$$

and, since the second member of G_x is zero, *q.p.*, we have $\int R\eta dS = 0$. Hence the resultant normal reaction passes through a point on the axis of x.

To determine the motion of the centre of inertia,

$$m\frac{du}{dt} = \Sigma X = -\mu \int RdS = -\mu mg;$$

therefore $$u = U - \mu gt.$$

Again $$m\frac{dv}{dt} = \Sigma Y = -\mu\frac{v}{u}\int RdS - \mu\frac{\omega}{u}\int R\xi dS = -\mu mg\frac{v}{u}, \qquad q.p.$$

hence $v = cu$; and since $v = 0$ when $u = U$, $c = 0$, therefore $v = 0$.

To find the angular velocity,

$$mk^2\frac{d\omega}{dt} = -\mu\frac{\omega}{u}\int R\xi^2 dS = -\mu K\frac{\omega}{u}\int \xi^2 dS; \qquad q.p.$$

but γ being the radius of gyration of the plane face, $\int \xi^2 dS = S\gamma^2$, and

$$mk^2\frac{d\omega}{dt} = -\mu mg\gamma^2\frac{\omega}{u}; \qquad q.p.$$

therefore $$\omega = \Omega\left(\frac{u}{U}\right)^{\frac{\gamma^2}{k^2}}.$$

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